Distributed Parameter Systems

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March 2, 2005

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Contents

The extra numbers of theorems and examples refer to the corresonding theorem or example in the book [CZ95]. In many cases the proofs given in [CZ95] are not repeated in these notes, but should be looked up directly from [CZ95]. Many of the results and proofs are also found in [Sta05].

Chapter 1

Background

1.1 Introduction

Many results in modern control theory are based on the notion of a *state space*, the purpose of which is to store enough information about the *past history* so that, if we know both *the present state and the future input*, then we can compute the *future output* of the system. The standard model in the linear case is a *system of ordinary differential equations*

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), & t \geq 0, \\ y(t) = Cz(t) + Du(t), & t \geq 0, \\ z(0) = z_0. \end{cases} \tag{1.1}$$

Here

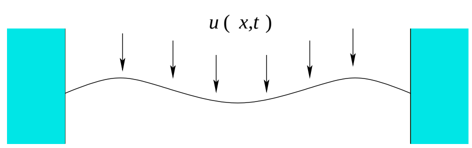
$$\begin{aligned} u(t) &= \text{the input } \in U = \mathbb{R}^m, \\ y(t) &= \text{the output } \in Y = \mathbb{R}^k, \\ z(t) &= \text{the state } \in Z = \mathbb{R}^n, \\ z_0 &= \text{the initial state.} \end{aligned}$$

A, B, C and D are matrices:

$$\begin{aligned} A &\in \mathbb{R}^{n\times n} \text{ is the state transition matrix,} \\ B &\in \mathbb{R}^{n\times m} \text{ the control matrix,} \\ C &\in \mathbb{R}^{k\times n} \text{ the observation matrix and} \\ D &\in \mathbb{R}^{k\times m} \text{ the feedthrough matrix} \end{aligned}$$

(Often $D = 0$) This approach is fine if the dynamic is simple, so that it can be appropriately described by a system of ordinary differential equations. However, certain systems can have a *very complicated behavior* (from a mathematical point of view). Typical such systems are described by *partial differential equations* or by *delay equations*. These systems are called **distributed parameter systems**: Their state is not described by a finite number of state variables, but by "infinitely many variables", such as the "velocity of the fluid" at infinitely many points. ("The variables are distributed throughout the fluid.")

Purpose of this course: To describe (some of) the mathematics which is needed to extend the "standard" theory for the ODE-system (1.1) to such more complicated systems. Formally we still write a system like (1.1), but now A, B, C and D can be *unbounded operators*, e.g. A is often a partial differential operator. This course is followed by another course on *transfer functions* where we describe a system in terms of its input-output behavior instead of "state space description". Both of these points of view are used side by side in the literature.



Example 1.1.1 (1.1.1) A guitar string.

$z(t, x) =$ the (vertical) deflection of the string at the point x at time t.
$u(x, t) =$ external force acting at the string (the wind blowing).

$$\text{(PDE)} \quad \underbrace{\rho \frac{\partial^2 z(x,t)}{\partial t^2}}_{\text{mass} \times \text{accel.}} - \underbrace{\alpha \frac{\partial^2 z(x,t)}{\partial x^2}}_{\substack{\text{restoring force} \\ \text{due to curvature}}} = \underbrace{u(x,t)}_{\text{force}}$$

$$\text{(BC)} \quad z(0,t) = 0,\ z(1,t) = 0 \text{ (fixed end points)}$$

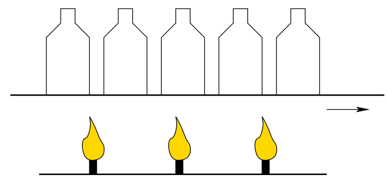
$$\text{(IC)} \quad z(x,0) = z_0(x),\ \frac{\partial}{\partial t} z(x,0) = z_1(x) \text{ (initial position and velocity)}$$

One possible problem: Try to choose $u(x, t)$ in such a way that $z(x, T) \equiv 0$ (for all x) at time T, i.e., *try to kill the motion of the string in a finite time.* Another problem: Instead of blowing at the string, let us move the end point.

Replace the right hand side in (PDE) by zero and replace (BC) by

$$z(0,t) = 0,\ z(1,t) = u(t).$$

Can we now choose $u(t)$ so that the motion of the string is killed in finite time?



Example 1.1.2 (1.1.3) *Pasteurization of beer.* Beer is pasturized. Bottles are transported on a belt through a oven. The temperature in the bottle should get high enough to Paseurize the beer, but not so high that it distroys the tast.

$$\text{(PDE)}\quad \underbrace{\frac{\partial z}{\partial t}(x,t)}_{\text{temp. rise}} = \underbrace{\mu\frac{\partial^2 z}{\partial x^2}(x,t)}_{\text{heat diffusion}} + \underbrace{v(t)\frac{\partial z}{\partial x}(x,t)}_{\text{belt is moving}} - \sigma[\ \underbrace{z(x,t)}_{\text{bottle temp.}} - \underbrace{u(x,t)}_{\text{temp. of furnace}}\ ]$$

and suitable (BC) and (IC). Control objective: We want the temperature $z(x,t)$ to be just right (for pasteurization).

See [CZ95] for additional examples.

1.2 Finite Dimensional Systems

By a finite dimensional system we mean the ODE case discussed on p. 1.

$$\begin{cases} \dot z(t) &= A\,z(t) + B\,u(t), \\ y(t) &= C\,z(t) + D\,u(t), \\ z(0) &= z_0. \end{cases}$$

By standard ODE-courses:

$$\begin{cases} z(t) = e^{At}z_0 + \int_0^t e^{A(t-s)}B\,u(s)\,ds, & t \ge 0, \\ y(t) = Ce^{At}z_0 + \int_0^t Ce^{A(t-s)}B\,u(s)\,ds + D\,u(t), & t \ge 0, \end{cases} \tag{1.2}$$

If $z_0 = 0$ (the system is at rest at time $t = 0$), then

$$y(t) = \int_0^t Ce^{A(t-s)} B\, u(s)\, ds + D\, u(t). \tag{1.3}$$

Definition 1.2.1 The distribution $D\delta + Ce^{At}B$ (where δ is the "δ-function", Dirac's delta) is the **impulse response** of the system.

(If $D = 0$ then this is an ordinary matrix-valued function, dimension $k \times m$.) By taking Laplace-transforms in (1.3) we get

$$\hat{y}(s) = \left[C(sI - A)^{-1}B + D\right]\, \hat{u}(s).$$

Definition 1.2.2 (1.2.1) The **transfer function** (överföringsfunktion, siirtofunktio) of the above system is

$$C(sI - A)^{-1}B + D.$$

The next course "Transfer Function Theory" will discuss such functions in detail without using this "(A, B, C, D)-representation".

Notation 1.2.3 We denote the system above by $\Sigma(A, B, C, D)$. We write $\Sigma(A, B, -)$ if C and D are irrelevant, and $\Sigma(A, -, C)$ if B and D are irrelevant.

Definition 1.2.4 (1.2.2) $\Sigma(A, B, -)$ is **controllable** (styrbar, ohjattava) if there is some $\tau > 0$ such that, given any $z_1 \in Z$ it is possible to find some control u such that, if $z_0 = 0$ and z is the solution of (1.1) with $z(0) = z_0 = 0$, then $z(\tau) = z_1$.

In other words, "every z_1 is reachable from the zero state in finite time".

From (1.2) we get the following lemma.

Lemma 1.2.5 $\Sigma(A, B-)$ *is controllable iff there is a* $\tau > 0$ *such that for each* $z \in Z$ *we can find a* u *such that*

$$\int_0^\tau e^{A(\tau-s)} B\, u(s)\, ds = z.$$

Problem 1.2.6 Which class of controls u should we allow in Definition 1.2.4 and Lemma 1.2.5?

There are two "standard" solutions.

(i) *Traditionally* (the ODE case) people are used to taking u to be continuous, or "piecewise continuous". This is actually OK in this course, because we will not have the time to go into the really "difficult" cases (boundary control of PDE).

(ii) In the *modern theory* one usually takes $u \in L^2([0,\tau];\mathbb{R}^m)$. This needs more background knowledge (Lebesgue integration theory), but instead it leads to much simpler computations since we can use the inner product in L^2 to simplify many results. (Much of the course "Transfer Function Theory" is based on L^2-theory).

In the following it does not matter (at this point) if we require $u \in C([0,\tau];\mathbb{R}^m)$ (u continuous) or $u \in L^2([0,\tau];\mathbb{R}^m)$ (u in L^2).

We denote

$$\mathcal{B}^\tau := \int_0^\tau e^{A(\tau-s)} B\, u(s)\, ds.$$

Then $\mathcal{B}^\tau$ maps $C([0,\tau];\mathbb{R}^m)$ into $Z = \mathbb{R}^n$, (and also $L^2([0,\tau];\mathbb{R}^m)$ into Z), and

$$\boxed{\Sigma(A,B,-) \text{ is controllable } \iff \text{ The range of } \mathcal{B}^\tau \text{ is all of } Z.}$$

(We call $\mathcal{B}^\tau$ the *controllability map* or *reachability map* over the time interval $[0,\tau]$.)

We shall often instead use the alternative *controllability map* or *reachability map* $\mathcal{B}_\tau$ over the time interval $[-\tau,0]$ given by

$$\mathcal{B}_\tau := \int_{-\tau}^0 e^{-As} B\, u(s)\, ds.$$

This is the map from the input u to the final state $x(0)$ if we take the initial time to be $-\tau$ and the initial state $x(-\tau)$ to be zero. It has the same range as $\mathcal{B}^\tau$, since $\mathcal{B}_\tau u$ is the same thing as $\mathcal{B}^\tau$ applied to a function u which has been shifted τ units to the right (simply use a change of integration variable in the definition of $\mathcal{B}_\tau$).

Definition 1.2.7 (1.2.2) The system $\Sigma(A, -, C)$ is **observable** (observerbar, tarkkailtava) if there is some $\tau > 0$ such that there is no initial condition $z_0 \neq 0$ for which the output function y of (1.1) (with input $u \equiv 0$) satisfies $y(t) = 0$ for all $t \in [0, \tau]$.

Thus, observable means that every $z_0 \neq 0$ is "visible in the output" on the interval $[0, \tau]$. We can reformulate this: Let $\mathcal{C}^\tau$ be the operator which maps $z \in \mathbb{R}^n$ into the corresponding output y, restricted to $[0, \tau]$. Thus, by (1.2),

$$(\mathcal{C}^\tau z)(t) = Ce^{At}z, \ 0 \leq t \leq \tau.$$

Then

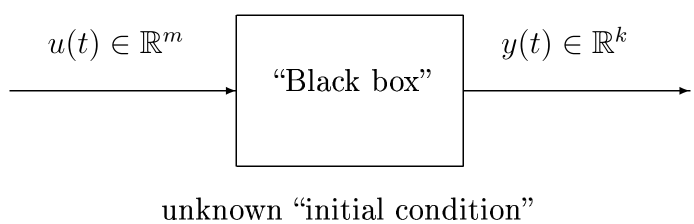
$$\boxed{\Sigma(A, -, C) \text{ is observable } \iff \mathcal{C}^\tau \text{ is one-to-one.}}$$

Definition 1.2.8 A is (exponentially) **stable** if

$$\|e^{At}\| \leq Me^{-\alpha t}, \ t \geq 0,$$

for some $M > 0, \ \alpha > 0$.

Realizations 1.2.9 In many practical situations there is no obvious state space, we only know that if we input certain signals u, then we get some other signals y:



The only thing which we can measure at least approximately, is the impulse response and the transfer function. Many MATLAB programs need a state space and matrices (A, B, C, D) to function.

Solution: Construct an "artificial" state space Z, and find (A, B, C, D) so that the transfer function of this system is approximately the measured transfer function.

Definition 1.2.10 $\Sigma(A, B, C, D)$ is a **minimal realization** of a given transfer function $G(s)$ if $\Sigma(A, B, -)$ is controllable, $\Sigma(A, -, C)$ is observable and

$$G(s) = C(sI - A)^{-1}B + D$$

(i.e., G is the transfer function of this system).

Theorem 1.2.11 *All minimal realizations of a given transfer function are similar to each other, i.e., if $\Sigma(A, B, C, D)$ and $\Sigma(A', B', C', D')$ are two realizations of the same transfer function, then there is an invertible matrix $E \in \mathbb{R}^{n \times n}$ such that*

$$\begin{aligned} A' &= E^{-1}AE, \\ B' &= E^{-1}B, \\ C' &= CE, \\ D' &= D \end{aligned}$$

Mathematically this corresponds to a simple *change of coordinates* in the state space Z.

1.3 Outline of this Course

The *preliminary plan* is to cover selected parts of [CZ95], chapters 1 - 4.

I. Intro. [CZ95], Chapter 1, this is what we have been doing so far.

II. Semigroup Theory. Describes how the state evolves in the state space when there is no input. We also ignore the output. Most of sections 2.1, 2.2 and 2.5 in [CZ95]. An alternative source is [Paz83].

III. The Cauchy Problem. Describes the evolution of the state z when the input u is nonzero. Section 3.1 and Theorem 3.2.1 in [CZ95].

IV. Inputs and Outputs. The "full" picture when we have input, state and output. Appropriate parts of chapter 4 in [CZ95].

V. Stability and Detectability. Only if time permits. Found in chapter 5 of [CZ95].

Restriction. Throughout this course B and C are *bounded operators* (the most interesting applications to boundary control require unbounded B and C).

Chapter 2

Semigroups (halvgrupp, puoliryhmä)

2.1 Needed Background

Definition 2.1.1 A **separable Hilbert space** is a (infinite-dimensional) complete inner product space.

Typical examples:

$$\begin{aligned}
L^2([0,\tau];\mathbb{C}^n) &= \Big\{\text{measurable } \mathbb{C}^n\text{-valued functions on } [0,\tau] \text{ with} \\
&\qquad \int_0^\tau |f(t)|^2\,dt < \infty\Big\} \\
\mathcal{W}^{1,2}([0,\tau];\mathbb{C}^n) &= \Big\{f \in L^2([0,\tau];\mathbb{C}^n) \text{ such that also } f' \in L^2([0,\tau];\mathbb{C}^n)\Big\} \\
\ell^2(\mathbb{N};\mathbb{C}^n) &= \Big\{\{a_k\}_{k=1}^\infty,\ \text{where each } a_k \in \mathbb{C}^n \text{ and} \\
&\qquad \sum_{k=1}^\infty \|a_k\|^2 < \infty\Big\}.
\end{aligned}$$

2.1.2 Crash Course: Roughly same as $\mathbb{C}^n$ with a "very large n", but some familiar convergence properties does not hold. For example, if $z_n \in \mathbb{Z}$ and $\|z_n\| \leq 1$ for all n, then it is still possible that no subsequence of $\{z_n\}$ converges to a limit. The other exceptions are similar.

2.2 An Abstract Differential Equation

Consider the "abstract differential equation" (1.1)

$$\begin{cases} \dot{z}(t) = A\,z(t) + B\,u(t), & t \geq 0, \\ y(t) = C\,z(t) + D\,u(t), & t \geq 0, \\ z(0) = z_0. \end{cases}$$

Here

$u(t) \in U$	(a Hilbert space, often $\mathbb{C}^m$),
$y(t) \in Y$	(another Hilbert space, often $\mathbb{C}^k$),
$z(t) \in Z$	(another Hilbert space, infinite dimensional if we have a DPS = Distributed Parameter System).

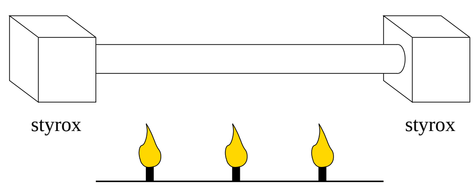
We have

$B \in \mathcal{L}(U;Z)$	(bounded linear operator),
$C \in \mathcal{L}(Z;Y)$	(bounded linear operator),
$B \in \mathcal{L}(U;Y)$	(bounded linear operator),

but A is unbounded. It is defined only on some subset $\mathscr{D}(A)$ of Z (the domain (definitionsmängd, määrittelyalue)) and it maps $\mathscr{D}(A)$ into Z. Thus, we have an implicit requirement on (1.1): For $A\,z(t)$ to be defined we must have

$$z(t) \in \mathscr{D}(A) \text{ for all } t \geq 0$$

(in particular, $z_0 \in \mathscr{D}(A)$).



Example 2.2.1 (2.1.1) Heated rod.
$z(x,t)$ = temperature
$u(x,t)$ = heat supply

(PDE) $$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial t^2}(x,t) + u(x,t)$$

(IC) $z(x,0) = \xi(x)$ (initial temperature)

(BC) $$\frac{\partial z}{\partial x}(0,t) = 0 = \frac{\partial z}{\partial x}(1,t) \text{ (isolated)}$$

Classical solution: We require that all the derivatives listed in (PDE) exist (in the sense taught in the basic calculus course) and are continuous. There is no difficulty in interpreting what the equation means.

Modern solution: We use "Hilbert space techniques" (and distribution theory) and require the equations to hold in the "L^2-sense".

$$Z = \text{ state space } = L^2([0,1];\mathbb{R}) = L^2(0,1)$$

The operator

$$A\,z = \frac{d^2 z}{dx^2}$$

is not defined for all $z \in Z$, only for some $z \in \mathscr{D}(A)$. The minimal requirement on $\mathscr{D}(A)$ is that "$z \in Z$ and $A\,z \in Z$", i.e.,

$$\frac{d^2 z}{dx^2} \in L^2(0,1),$$

so z should be "two times differentiable in the L^2-sense".

Definition 2.2.2 $\mathcal{W}^{1,2}(0,1)$ is the set of functions $z : [0,1] \mapsto \mathbb{R}$ which are of the form

$$z(t) = z_0 + \int_0^t w(s)\; ds$$

for some $z_0 \in \mathbb{R}$ and some $w \in L^2(0,1)$.

Definition 2.2.3 We call the function w in Definition 2.2.2 either the **L^2-derivative of** z or the **distribution derivative of** z, and denote $w = z'$. Thus, for $z \in \mathcal{W}^{1,2}(0,1)$, we have $z(t) = z_0 + \int_0^t z'(s)\; ds$, where $z' \in L^2$.

Definition 2.2.4 $\mathcal{W}^{2,2}(0,1) =$ the set of functions $z \in C^1[0,1]$ whose derivative z' satisfies $z' \in \mathcal{W}^{1,2}(0,1)$.

Back to the example. In order to "make sense" out of the term $\dfrac{\partial^2 z}{\partial x^2}$ in (PDE) we require (at least) that for each fixed $t \geq 0$, the function $x \mapsto z(x,t)$ belongs to $\mathcal{W}^{2,2}(0,1)$. (We shall add some other conditions, related to the boundary conditions).

Boundary condition: For each fixed t, if $x \mapsto z(x,t) \in \mathcal{W}^{2,2}(0,1)$, then z' is continuous in x, so $z'(0,t)$ and $z'(1,t)$ are well defined. The condition (BC) require these to be zero. Therefore we have

$$\boxed{\mathscr{D}(A) = \{z \in \mathcal{W}^{2,2}(0,1) \mid z'(0) = 0 = z'(1)\}.}$$

Note. Here we had "Neumann boundary condition". If we instead had "Dirichlet boundary condition", then the appropriate $\mathscr{D}(A)$ would have been

$$\mathcal{W}_0^{2,2}(0,1) = \{z \in \mathcal{W}^{2,2}(0,1) \mid z(0) = 0 = z(1)\}.$$

There are many other possibile configurations. Here we had

$$A\,z = \frac{d^2 z}{dx^2},$$

$$\mathscr{D}(A) = \text{ as above,}$$

$$B = 1 = \text{ identity operator} \Rightarrow \text{ (at least formally)}$$

$$\begin{cases} z'(t) = A\,z(t) + B\,u(t), & t \geq 0 \\ z(0) = z_0 \in \mathscr{D}(A) & \text{since } z(t) \in \mathscr{D}(A) \text{ for all } t \geq 0. \end{cases}$$

Hot question: What if we instead used boundary control:

$$z(1,t) = w(t) = \text{ the control.}$$

This may force $z(t)$ to leave $\mathscr{D}(A)$, and we run into technical problems (need "unbounded" B). Not covered in this course.

2.2.5 Solution of Example 2.2.1. We separate the variables. Try solutions of the type

$$z(x,t) = e^{\lambda t} v(x).$$

Substitute into equation to get

$$\underbrace{\lambda e^{\lambda t} v(x) = e^{\lambda t} v''(x)}_{\text{homogeneous part}} + \underbrace{\quad\cdots\quad}_{\text{doesn't depend on } x}.$$

Taking also (BC) in account we get an eigenvalue problem (egenvärdeproblem, ominaisarvotehtävä)

$$v''(x) - \lambda v(x) = 0;\; v'(0) = 0 = v'(1).$$

The only possible solutions are

$$v(x) = \text{constant} \cdot \cos(\sqrt{-\lambda}\,x),$$

where

$$\sqrt{-\lambda} = n\pi, \text{ i.e., } \lambda = -n^2\pi^2,\; n = 0, 1, 2, \ldots$$

Let us normalize v so that

$$\|v\|^2 = \int_0^1 |v(x)|^2 \, dx = 1.$$

This leads to

$$v_n(x) = \begin{cases} 1 & , \ n = 0, \\ \sqrt{2}\cos(\pi n x) & , \ n = 1, 2, 3, \dots \end{cases}$$

Project $z(x,t)$ onto these eigenfunctions to get

$$z(x,t) = \sum_{n=0}^{\infty} z_n(t) v_n(x),$$

where

$$z_n(t) = \langle z, v_n \rangle = \int_0^1 z(x,t) v_n(x) \, dx.$$

So far all that has been said above is mathematically correct. Since the purpose of this discussion is to motivate what is coming up next, let us proceed formally (without verifying that all the tricks we do are permitted). However, it is possible to make the following, too, exact.

Take the inner product of (PDE) with v_n, and integrate by parts (the boundary terms vanish because of (BC)):

$$\langle z_t, v_n \rangle = \langle z_{xx}, v_n \rangle + \langle u, v_n \rangle.$$

This gives for $n = 0$

$$\begin{aligned} & \int_0^1 z_t \, dx = \int_0^1 z_{xx} \, dx + \int_0^1 u(x,t) \, dt \\ \overset{?}{\iff} \quad & \frac{d}{dt} \underbrace{\int_0^1 z(t,x) \, dx}_{=z_0(t)} = \underbrace{z_x(1,t) - z_x(0,1)}_{=0} + \int_0^1 u(x,t) \, dx \\ \iff \quad & \dot{z}_0 = \int_0^1 u(x,t) \, dx. \end{aligned}$$

Let us project also u on the same eigenfunctions:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t) v_n(x);$$

where

$$u_n(t) = \langle u, v_n \rangle = \int_0^1 u(x,t)v_n(x)\,dx.$$

Then the above equation becomes

$$\dot{z}_0(t) = u_0(t),\ t \geq 0.$$

Since $z_0(0) = \int_0^1 z(x,0)\,dx = \int_0^1 \xi(x)\,dx$ (here we used (IC)), we can write this explicitly as

$$z_0(t) = z_0(0) + \int_0^t u_0(s)\,ds,$$

where

$$z_0(0) = \int_0^1 \xi(x)\,dx \quad \text{and} \quad u_0(s) = \int_0^1 u(x,s)\,dx.$$

For $n \geq 1$ we get a similar result (remember to integrate by parts):

$$\dot{z}_n(t) = -(n\pi)^2 z_n(t) + u_n(t),$$

with

$$z_n(0) = \int_0^1 z(x,0)v_n(x)\,dx \quad \text{and} \quad u_n(t) = \int_0^1 u(x,t)v_n(x)\,dx,$$

which can be solved in the same way (by the variation of constants formula):

$$z_n(t) = e^{-(n\pi)^2 t} z_n(0) + \int_0^t e^{-(n\pi)^2(t-s)} u_n(s)\,ds.$$

(Note that the case $n = 0$ is a special case of this!) Adding the terms we get (at least formally),

$$\begin{aligned}
z(x,t) = &\quad \sum_{n=0}^{\infty} z_n(t) v_n(x) \\
\overset{?}{=} &\quad \sum_{n=0}^{\infty} e^{-(n\pi)^2 t} v_n(x) z_n(0) + \int_0^t \sum_{n=0}^{\infty} e^{-(n\pi)^2 (t-s)} v_n(x) u_n(s)\, ds \\
= &\quad \sum_{n=0}^{\infty} e^{-(n\pi)^2 t} v_n(x) \int_0^1 v_n(y)\xi(y)\, dy \\
&\quad + \int_0^t \sum_{n=0}^{\infty} e^{-(n\pi)^2 (t-s)} \int_0^1 v_n(y) v(s,y)\, dy\, ds \\
\overset{?}{=} &\quad \int_0^1 \left[\sum_{n=0}^{\infty} e^{-(n\pi)^2 t} v_n(x) v_n(y) \right] \xi(y)\, dy \\
&\quad + \int_0^t \left[\int_0^1 \sum_{n=0}^{\infty} e^{-(n\pi)^2 (t-s)} v_n(x) v_n(y) \right] u(s,y)\, dy\, ds \\
= &\quad \int_0^1 g(t,x,y)\xi(y)\, dy + \int_0^t \int_0^1 g(t-s,x,y) u(s,y)\, dy\, ds,
\end{aligned}$$

where

$$g(t,x,y) = \sum_{n=0}^{\infty} e^{-(n\pi)^2 t} v_n(x) v_n(y) = 1 + 2\sum_{n=0}^{\infty} e^{-(n\pi)^2 t} \cos(n\pi x)\cos(n\pi y)$$

is the **Green's function**. Once more: We get

$$z(x,t) = \int_0^1 g(t,x,y)\xi(y)\, dy + \int_0^t \int_0^1 g(t-s,x,y) u(s,y)\, dy\, ds.$$

Interpretation: Compare this to the "classical" formula:

$$z(t) = e^{At} z(0) + \int_0^t e^{A(t-s)} u(s)\, ds.$$

Let us define the operator T^t:

$$(T^t \xi)(x) := \int_0^1 g(t,x,y)\xi(y)\, dy.$$

Then, at least formally, the solution of Example 2.2.1 is given by

$$z(t) = T^t \xi + \int_0^t T^{(t-s)} u(s)\, ds.$$

The operator T^t acts on functions in $Z = L^2(0,1)$, and it can be shown that it maps these functions into continuous functions, and even $T^t \subset \mathcal{L}(Z)$ (bounded linear operator $Z \mapsto Z$). Thus,

we get a family of operators $T^t \in \mathcal{L}(Z)$ which act like the standard "fundamental solution" e^{At} of the system of ODE:s $\dot{z} = A\,z$.

2.3 Properties of Fundamental Solution

Let us assume that the solution of the "differential equation"

$$\dot{z}(t) = A\,z(t); \; z(0) = z_0 \tag{2.1}$$

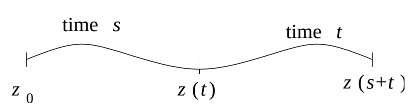
can be written in the form

$$z(t) = T^t z_0,$$

for some family of operators T^t (as above). What kind of properties does T^t have? (Formally, $T^t = e^{At}$.)

(i) $\underline{T^0 = 1}$ (because $T^0 z_0 = z(0) = z_0$ for all $z_0 \in Z$)

(ii) $\underline{T^{(s+t)} = T^s T^t}$ for the following reason. Start at the point z_0, solve (2.1) to get $x(t)$, start again from the point $x(t)$ and let (2.1) evolve for s more time units. (Total time $= t + s$.)



If the system is *time-independent*, then it does not matter if we start at time t and go to time $t+s$ or if we start at time zero and go to time s, as long as we use the same initial state $z(t)$ in both cases. Thus,

$$\begin{aligned} T^{(t+s)} z_0 = z(t+s) = \quad & \text{"solve equation (2.1) for } s \text{ time units} \\ & \text{with initial state } z(t)\text{"} \\ & = T^s z(t) \\ & = T^s T^t z_0, \end{aligned}$$

so $T^{(s+t)} z_0 = T^s T^t z_0$ for all $z_0 \in Z$.

(iii) Continuity: We would like $x(t)$ to be continuously differentiable (so that (2.1) makes sense), but we are also willing to settle for solutions which are only continuous in Z (then (2.1) must be interpreted in some "distributional sense"). This means that for each $z_0 \in Z$, the function $t \mapsto T^t z_0$ $(= z(t))$ is continuous with values in Z. Called *strong continuity.*

2.4 Different Continuity Notions

Notation 2.4.1 $\mathcal{L}(Z)$ = The set of all bounded linear operators $Z \mapsto Z$.

$$\|T\|_{\mathcal{L}(Z)} = \sup_{z \in Z} \frac{\|Tz\|}{\|z\|} = \sup_{\|z\| \leq 1} \|Tz\|.$$

Definition 2.4.2 The statement $T_n \to T$ as $n \to \infty$ can be interpreted in many ways (here $T_n,\ T \in \mathcal{L}(Z)$).

(i) $T_n \to T$ **uniformly** (likformigt, tasaisesti) or "in operator norm" if

$$\|T_n - T\|_{\mathcal{L}(Z)} \to 0 \text{ as } n \to \infty.$$

Equivalently $\lim_{n\to\infty} \sup_{|z|\leq 1} \|T_n z - Tz\|_Z = 0$.

(ii) $T_n \to T$ **strongly** (starkt, vahvasti) if $T_n z \to Tz$ for every fixed $z \in Z$, or equivalently $\lim_{n\to\infty} \|T_n z - Tz\| = 0$ for each fixed $z \in Z$. Also called **pointwise convergence**.

(iii) $T_n \to T$ **weakly** (svagt, heikosti) if for all $z_1, z_2 \in Z$,

$$\langle z_1, T_n z_2 \rangle \to \langle z_1, T z_2 \rangle \text{ as } n \to \infty.$$

We use the same concept to define different types of continuity of an operator-valued function.

Definition 2.4.3 **(i)** T^t is a **uniformly continuous** function of t (**continuous in norm**) at a point t_0 if

$$\lim_{t \to t_0} \|T^t - T^{t_0}\|_{\mathcal{L}(Z)} = 0.$$

(ii) T^t is a **strongly continuous** function of t at a point t_0 if

$$\lim_{t \to t_0} \|T^t z - T^{t_0} z\|_Z = 0$$

for each fixed $z \in Z$.

(i) T^t is a **weakly continuous** function of t at t_0 if

$$\lim_{t \to 0} \langle z_1, (T^t - T^{t_0}) z_2 \rangle = 0$$

for all fixed $z_1, z_2 \in Z$.

In this course the most used notion is strong continuity. This is the type of continuity that guarantees that *trajectories are continuous* (see previous section). Note: Weak continuity = strong continuity for semigroups (this is a highly nontrivial fact).

2.5 Strongly Continuous Semigroups

Definition 2.5.1 (2.1.2) A function T^t, defined for $t \in [0, \infty)$ with values in $\mathcal{L}(Z)$, is a **strongly continuous semigroup**, or C_0-semigroup, if the following conditions hold:

(i) $T^0 = 1$ (the identity operator)

(ii) $T^{(t+s)} = T^t T^s, \ t, s \geq 0$

(iii) for all $z \in Z$, $\lim_{t \downarrow 0} T^t z = z$ (strong continuity at zero).

Example 2.5.2 (2.1.3) Let $A \in \mathcal{L}(Z)$, and define

$$T^t = e^{At} = 1 + At + \frac{(At)^2}{2!} = + \cdots$$

This semigroup is even continuous in operator norm ("uniformly"). The series converges in the Banach space $\mathcal{L}(Z)$, uniformly on bounded subsets.

Example 2.5.3 (2.1.4) Left shift in $L^2(0, \infty)$:

$$(T^t h)(x) = h(t + x), \ t, x \geq 0.$$

Example 2.5.4 Right shift in $L^2(0,\infty)$:

$$(T^t h)(x) = \begin{cases} h(x-t) & \text{if } x \geq t \\ 0 & \text{if } 0 \leq x < t. \end{cases}$$

"Zero fill" (replace missing values by zero).

Example 2.5.5 Left shift in $L^2(-\infty,\infty)$:

$$(T^t h)(x) = h(x+t),\ t \geq 0,\ x \in \mathbb{R}.$$

Theorem 2.5.6 *(Improved 2.1.5) Let $\{\phi_n\}_{n=1}^\infty$ be an orthonormal basis in Z. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of complex numbers. Define*

$$T^t z = \sum_{n=1}^{\infty} e^{\lambda_n t} \langle z, \phi_n \rangle \phi_n.$$

This is a C_0-semigroup if and only if

$$\sup_{n\in\mathbb{N}} \Re(\lambda_n) < \infty.$$

PROOF. Homework (very similar to the proof of Example 2.1.5 in [CZ95]).

Note. Example 2.1.1 is of this type.

Comment 2.5.7 The *modern approach* to PDE:s of parabolic or hyperbolic type, or more generally equations which contain a time variable, is the following.

(a) Formulate the problem in the form

$$\dot{z}(t) = Az(t),\ z(0) = z_0,$$

and prove that you get a "fundamental solution", i.e., a semigroup.

(b) Study separately under what conditions the solution is "smooth enough in time and space" so that you get a classical solution (often easy).

Note. *Elliptic equations* require a different technique. These equations (eigenvalueproblems) appear also as "auxiliary problems" in the approach described above, they give stationary solutions (solutions which do not vary in time, so that $\dot{z} = 0$, i.e., $Az = 0$).

2.6 Properties of Semigroups

Theorem 2.6.1 *(2.1.6) All C_0-semigroups have the following properties:*

(a) $\sup_{0\le t\le M} \|T^t\| < \infty$ *(for each fixed M),*

(b) *$t \mapsto T^t z$ is continuous for all $t \ge 0$ and all $z \in Z$ (not just at $t = 0$) (says that trajectories are continuous),*

(c) $\dfrac{1}{t}\displaystyle\int_0^t T^s z \, ds \to z$ *as $t \to 0+$,*

(d) *the "growth bound" ω_0 of T satisfies*

$$\omega_0 \overset{\text{def}}{=} \inf_{t>0}\left(\frac{1}{t}\log\|T^t\|\right) = \lim_{t\to\infty}\frac{1}{t}\log\|T^t\| < \infty,$$

(e) $\|T^t\| \le M_\omega e^{\omega t}$ *for each $\omega > \omega_0$ (M_ω depends on ω).*

Proof. See [CZ95] or [Sta05].

How is all of this related to the equation

$$\dot z(t) = Az(t),\ t \ge 0,\ z(0) = z_0?$$

In the classical case where A is a matrix and $T^t = e^{At}$ we can recover A from T:

$$Az_0 = \dot z(0) = \lim_{t\to 0}\frac{1}{t}(z(t) - z(0)) = \lim_{t\to 0}\frac{1}{t}(T^t z_0 - z_0) = \dot T^0 z_0,$$

So $A = \dot T^0$. The same idea works more generally:

Definition 2.6.2 (2.1.8) the **generator** A of T is defined for those $z \in Z$ for which $T^t z$ is differentiable at zero and we define it by

$$Az = \lim_{h\to 0+}\frac{1}{h}(T^h - 1)z$$

(with $\mathscr{D}(A) = \{z \in Z \mid \text{the above limit exists}\}$).

Here $\mathscr{D}(A)$ is the domain (definitionsmängd, määrittelyalue). It is easily seen to be a subspace of Z (i.e., a vector space). The above limit is computed in the sense of "norm-limit" in Z, i.e., we require that

$$\boxed{\lim_{t\downarrow 0}\left\|\frac{1}{t}(T^t-1)z-Az\right\|_Z = 0 \text{ for all } z\in\mathscr{D}(A).}$$

Usually $\mathscr{D}(A)\neq Z$ (in all interesting cases). The only exception (less interesting) is:

Example 2.6.3 if $T^t=e^{At}$ where $A\in\mathcal{L}(Z)$, then the generator of T is A, and $\mathscr{D}(A)=Z$.

PROOF. See [CZ95] or [Sta05].

Theorem 2.6.4 *(2.1.10) Let A be the generator of a C_0-semigroup T.*

(a) *$\mathscr{D}(A)$ is invariant under T: If $z_0\in\mathscr{D}(A)$, then $T^tz_0\in\mathscr{D}(A)$ for all $t\geq 0$.*

(b) *For $z_0\in\mathscr{D}(A)$, the trajectory (bana, rata) $t\mapsto z(t)=T^tz_0$ is differentiable and*
$$\frac{d}{dt}T^tz_0=AT^tz_0=T^tAz_0,\ t\geq 0$$
(thus, if T^tz_0 is differentiable at zero, then it is differentiable for all $t\geq 0$).

(c) *If $z_0\in\mathscr{D}(A^n)$ (this means that $z_0\in\mathscr{D}(A)$, $Az_0\in\mathscr{D}(A)$, $A(Az_0)\in\mathscr{D}(A)$ etc. up to $\underbrace{A(A(A\ldots}_{n-1\text{ times}}z))\in\mathscr{D}(A)$), then*
$$(\frac{d}{dt})^nT^tz_0=A^nT^tz_0=T^tA^nz_0=A^kT^tA^{n-k}z_0,\ 0\leq k\leq n,\ t\geq 0.$$

(d) $T^tz_0-z_0=\displaystyle\int_0^t T^sAz_0\,ds,\ z_0\in\mathscr{D}(A),\ t\geq 0.$

(e) *$\int_0^t T^sz_0\,ds\in\mathscr{D}(A)$ for all $z_0\in Z$ and*
$$A\int_0^t T^sz_0\,ds=T^tz_0-z_0,\ t\geq 0.$$
Moreover, $\mathscr{D}(A)$ is dense in Z ($\overline{\mathscr{D}(A)}=Z$).

(f) *A is a closed operator (explained below).*

(g) $\bigcap_{n=1}^{\infty} \mathscr{D}(A)$ *is dense in* Z*, and* $T^t z_0 \in \mathcal{C}^\infty$ *if* $z_0 \in \bigcap_{n=1}^{\infty} \mathscr{D}(A^n)$*.*

Comments: (e): "Dense" means that the closure of $\mathscr{D}(A)$ is all of Z.

(f): "A closed" means that "the graph of A is closed". By the graph of A we mean the set of pairs

$$\begin{pmatrix} Az \\ z \end{pmatrix} \in Z \times Z,$$

where z varies over $\mathscr{D}(A)$. If we use the standard norm in Z^2,

$$\left\| \begin{pmatrix} y \\ z \end{pmatrix} \right\| = \left(\|y\|_Z^2 + \|z\|_Z^2 \right)^{1/2},$$

then a subset is closed iff the limit of every sequence of points in this set belongs to the set: Denote

$$G(A) = \left\{ \begin{pmatrix} Az \\ z \end{pmatrix} \in Z^2 \;\middle|\; z \in \mathscr{D}(A) \right\} = \text{the graph of } A.$$

Take a sequence of points $\begin{pmatrix} y_n \\ z_n \end{pmatrix} \in G(A)$, which converges to $\begin{pmatrix} y \\ z \end{pmatrix} \in Z^2$. The condition $\begin{pmatrix} y_n \\ z_n \end{pmatrix} \in G(A)$ means that $z_n \in \mathscr{D}(A)$ and $y_n = Az_n$. Convergence means that $z_n \to z$ in Z and also $Az_n \to y$ in Z. The limit point belongs to $G(A)$ if and only if $z \in \mathscr{D}(A)$ and $y = Az$. Thus, "A closed" means

$$\boxed{\text{if } z_n \in \mathscr{D}(A),\ z_n \to z \text{ and } Az_n \to y, \text{ then } z \in \mathscr{D}(A) \text{ and } y = Az.}$$

This is important, learn by heart!

Proof of selected parts of Theorem 2.6.4.

(a),(b): Let $z \in \mathscr{D}(A)$. Then

$$\lim_{h \to 0+} \frac{1}{h}(T^h - 1)z = Az. \tag{2.2}$$

Now take $t > 0$. Then for $n > 0$:

$$\begin{aligned} \frac{1}{h}[T^{(t+h)} - T^t]z &= \frac{1}{h}[T^h T^t - T^t]z && (2.3) \\ &= \frac{1}{h}[T^h - 1]T^t z && (2.4) \\ &= \frac{1}{h}[T^t T^h - T^t]z && (2.5) \\ &= T^t \frac{1}{h}[T^h - 1]z. && (2.6) \end{aligned}$$

By (2.2), the last line (2.6) has a limit as $h \to 0+$. Therefore also all the other lines have limits as $h \to 0+$. Step (2.4) implies that $T^t z \in \mathscr{D}(A)$. Combining (2.3), (2.4) and (2.6) we get

$$\lim_{h \to 0+} \frac{1}{h}[T^{(t+h)} - T^t]z = AT^t z = T^t Az.$$

This proves differentiability from the right. Left-differentiability is slightly more complicated. For $0 < h < t$,

$$\begin{aligned} \frac{1}{-h}[T^{(t-h)} - T^t]z \text{ (as above)} &= \frac{1}{-h}[T^{(t-h)} - T^k T^{(t-h)}]z && (2.7) \\ &= \frac{1}{-h}[1 - T^h]T^{(t-h)}z && (2.8) \\ &= T^{(t-h)} \underbrace{\frac{1}{-h}[1 - T^h]z}_{\text{limit exists as } k \to 0+} . && (2.9) \end{aligned}$$

Also the full limit of (2.9) exists as $h \to 0+$ and so does the limit in (2.8). See homework.

(c): Use (b) and induction.

(d): Let $z \in \mathscr{D}(A) \Rightarrow \frac{d}{dt}T^t z = T^t Az$. (Integrate this function.)

$$T^t z - \underbrace{T^0 z}_{=z} = \int_0^t \frac{d}{ds} T^s z \, ds = \int_0^t T^s Az \, ds.$$

(e): Take $z \in Z$, $t > 0$. Then

$$\begin{aligned} \frac{1}{h}[T^h - 1]\int_0^t T^s z \, ds &= \int_0^t \frac{1}{h}[T^{s+h} - T^s]z \, ds \\ \text{(change int. var. } s + h \to w) &= \int_h^{t+h} \frac{1}{h} T^s z \, ds - \int_0^t \frac{1}{h} T^s z \, ds \\ &= \frac{1}{h}\int_h^{t+h} T^s z \, ds - \frac{1}{h}\int_0^t T^s z \, ds \\ &\to T^t z - z \text{ as } h \to 0+ . \end{aligned}$$

This gives $\int_0^t T^s z \, ds \in \mathscr{D}(A)$ and $A\int_0^t T^s z \, ds = T^t z - z$. That $\mathscr{D}(A)$ is dense in Z follows from the fact that for each $t > 0$,

$$\frac{1}{t}\int_0^t T^s z \, ds \in \mathscr{D}(A)$$

and

$$\frac{1}{t}\int_0^t T^s z\,ds \to z \text{ as } t \to 0+\,.$$

(f): See [CZ95] or [Sta05]. There is no need to use Lebesgue's dominated convergence here. A much simpler approach is sufficient (see extra homework).

(g): See [CZ95] or [Sta05]. $\square$

Corollary 2.6.5 *For a certain dense set of $z_0 \in Z$, the trajectories $t \to T^t z_0$ are $\mathcal{C}^\infty$ smooth.*

Note 2.6.6 *Think: $z \in \mathscr{D}(A)$ means often that z has one or more space-derivatives in Z. For example,*

$$Az = \frac{\partial^2}{\partial x^2} z, \text{ or } Az = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) z.$$

2.7 The Resolvent Operator

Since the generator A is unbounded, it is difficult to work with. It is often simpler to work with the inverse $(\lambda - A)^{-1}$ instead, for some suitable $\lambda \in \mathbb{C}$.

Definition 2.7.1 Let $A : \mathscr{D}(A) \mapsto X$ be a linear operator. A point $\lambda \in \mathbb{C}$ belongs to the **resolvent set** of A if $\lambda - A$ is one-to-one, its range is the whole space, and the inverse

$$R(\lambda, A) = (\lambda - A)^{-1}$$

is a bounded linear operator $X \mapsto X$. We denote the resolvent set by $\rho(A)$. The complement $\sigma(A)$ is called the **spectrum** of A.

Lemma 2.7.2 *Let $A : \mathscr{D}(A) \mapsto X$ be a linear operator (with $\mathscr{D}(A) \subset X$).*

(i) *If $\rho(A) \neq \varnothing$, then A is closed.*

(ii) *If A is closed, then $\lambda \in \mathbb{C}$ belongs to $\rho(A)$ if and only if $\lambda - A$ maps $\mathscr{D}(A)$ one-to-one onto X.*

Proof. Homework.

Lemma 2.7.3 *Resolvent identity. Let $A : X \supset \mathscr{D}(A) \mapsto X$ be a (closed) linear operator with $\rho(A) \neq \varnothing$. Then, for all $\lambda, \mu \in \rho(A)$*

$$(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1} = (\mu - \lambda)(\mu - A)^{-1}(\lambda - A)^{-1}.$$

Proof. See [Sta05], p. 72.

Notation. $1 =$ the identity operator.

Theorem 2.7.4 *(2.1.11) Let T be a $\mathcal{C}_0$-semigroup with growth bound ω_0 and generator A. Then $\rho(A)$ contains the half-plane $\mathbb{C}_{\omega_0} = \{\lambda \in \mathbb{C} \mid \Re\lambda > \omega_0\}$. Moreover, for every $\omega > \omega_0$ and every $\lambda \in \mathbb{C}_\omega = \{\lambda \in \mathbb{C} \mid \Re\lambda > \omega\}$, we have*

(a) $(\lambda - A)^{-1} z = \int_0^\infty e^{-\lambda t} T^t z \, dt, \; z \in Z,$

(b) $\|(\lambda - A)^{-1}\| \leq \frac{M}{\Re\lambda - \omega}$ *(M is a constant)*

(c) $\lim_{\lambda \to +\infty} \lambda(\lambda - A)^{-1} z = z, \; z \in Z$ *(λ is real).*

Proof. (a), (b): Define

$$R_\lambda z = \int_0^\infty e^{-zt} T^t z \, dt.$$

Let $\Re(\lambda) > \omega > \omega_0$. Recall that $\|T^t\| \leq M e^{\omega t}$ for some $M < \infty$, which implies that the integral converges, and

$$\begin{aligned} \|R_\lambda z\| &\leq \int_0^\infty \|e^{-\lambda t} T^t z\| \, dt \\ &= \int_0^\infty e^{-\Re\lambda t} \|T^t z\| \, dt \\ &\leq \int_0^\infty e^{-(\Re\lambda + \omega)t} M \, dt \, \|z\| \\ &= \frac{M}{\Re\lambda - \omega} \|z\|. \end{aligned}$$

This proves (b), if we can show that $R_\lambda = (\lambda - A)^{-1}$.

Claim: R_λ maps X into $\mathscr{D}(A)$.

Proof.

$$\begin{aligned}
\frac{1}{h}(T^h - 1)R_\lambda z &= \frac{1}{h}\int_0^\infty e^{-\lambda t}(T^{t+h}z - T^t z)\,dt \\
&= \frac{1}{h}\left[\int_h^\infty e^{-\lambda(t-h)}T^t z\,dt - \int_0^\infty e^{-\lambda t}T^t z\,dt\right] \\
&= -\frac{1}{h}\int_0^h e^{-\lambda(t-h)}T^t z\,dt + \frac{1}{h}(e^{\lambda h} - 1)\int_0^\infty e^{-\lambda t}T^t z\,dt \\
&\to -T^0 z + \lambda R_\lambda z,
\end{aligned}$$

so $R_\lambda z \in \mathscr{D}(A)$ and $AR_\lambda z = -z + \lambda R_\lambda z$ which implies that

$$(\lambda - A)R_\lambda z = z.$$

Thus, $(\lambda - A)$ is a left-inverse of R_λ. To prove that $\lambda - A$ is also a right-inverse we compute (for all $z \in \mathscr{D}(A)$)

$$\begin{aligned}
R_\lambda A z &= \int_0^\infty e^{-\lambda t}T^t A z\,dt \\
&= \int_0^\infty e^{-\lambda t}\big(\frac{d}{dt}T^t z\big)\,dt \text{ (integrate by parts)} \\
&= \big[e^{-\lambda t}T^t z\big]_0^\infty + \lambda\int_0^\infty e^{-\lambda t}T^t z\,dt \\
&= -z + \lambda R_\lambda z,
\end{aligned}$$

so $R_\lambda(\lambda - A)z = z$ for all $z \in \mathscr{D}(A)$. Thus, $(\lambda - A)$ is invertible, $\lambda \in \rho(A)$, and

$$(\lambda - A)^{-1} = R_\lambda.$$

(This proof is "easier" (more elementary) than the proof in [CZ95].)

(c): If $z \in \mathscr{D}(A)$ and $\lambda > \omega > \omega_0$ (λ real), then

$$\lambda(\lambda - A)^{-1}z - z = (\lambda - \lambda + A)(\lambda - A)^{-1}z = A(\lambda - A)^{-1}z = (\lambda - A)^{-1}Az,$$

and

$$\|\lambda(\lambda - A)^{-1}z - z\| \le \|(\lambda - A)^{-1}\|\|Az\| \le \frac{M}{\lambda - \omega}\|Az\| \to 0 \text{ as } \lambda \to +\infty.$$

If $z \in \mathscr{D}(A)$, then we choose $z_0 \in \mathscr{D}(A)$ with $\|z - z_0\| \le \varepsilon$. Then

$$\|\lambda(\lambda - A)^{-1}z - z\| \le \|\lambda(\lambda - A)^{-1}z_0 - z_0\| + \|\lambda(\lambda - A)^{-1}\|\|z - z_0\| + \|z - z_0\|$$

$$\leq \underbrace{\|\lambda(\lambda - A)^{-1}z_0 - z_0\|}_{\to 0} + \Big(\underbrace{\frac{\lambda M}{\lambda - \omega}}_{\to M} + 1\Big) \underbrace{\|z - z_0\|}_{\leq \varepsilon},$$

so $\limsup_{\lambda\to\infty}\|\lambda(\lambda - A)^{-1}z - z\| \leq (M+1)\varepsilon$. Since ε can be arbitrarily small,

$$\lim_{\lambda\to+\infty} \|\lambda(\lambda - A)^{-1}z - z\| = 0 \text{ for all } z \in Z.$$

$\square$

Note 2.7.5 Because of (c), we call

$$J_\lambda = \lambda(\lambda - A)^{-1}$$

the **Yoshida approximation** of the identity.

Lemma 2.7.6 *If T is a C_0-semigroup on Z and $\alpha \in \mathbb{C}$, then also $e^{\alpha t}T^t$ is a C_0 semigroup on Z. Its generator is $A + \alpha$, where A is the generator of T.*

Proof. Easy.

Thus, by using an exponential shift (see above) we can make the growth bound < 0 if we want. Sometimes useful.

2.8 The Hille-Yoshida Theorem

Theorem 2.8.1 *Hille-Yoshida.* *$A : Z \supset \mathscr{D}(A) \mapsto Z$ is the generator of a C_0-semigroup T on Z satisfying $\|T^t\| \leq Me^{\omega t}$, $t \geq 0$, if and only if*

(i) *$\mathscr{D}(A)$ is dense in Z*

(ii) *every real $\lambda > \omega$ belongs to $\rho(A)$, and for all $\lambda > \omega$ and $n = 1, 2, 3, \ldots$*

$$\|(\lambda - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}.$$

Note. If $\rho(A) \neq \varnothing$ then A is closed. (Lemma 2.7.2)

This theorem is very important, but since we are short of time, we skip most of the proofs. See [CZ95] or [Sta05] for the full proof. We only look at some results which come up in the proof and which are useful in other connections too. Recall the resolvent identity (Lemma 2.7.3).

Lemma 2.8.2 *The mapping $\alpha \mapsto (\alpha - A)^{-1}$ is an analytic function defined on $\rho(A)$ with values in $\mathcal{L}(Z)$, and for all $n = 1, 2, 3, \dots$*

$$\left(\frac{d}{d\lambda}\right)^n (\lambda - A)^{-1} z = \int_0^\infty (-t)^n e^{-\lambda t} T^t z \, dt = (-1)^n n! (\lambda - A)^{-n-1} z, \; z \in Z.$$

("Analytic" means that the complex derivative

$$\frac{d}{d\lambda}(\lambda - A)^{-1} = \lim_{h \to 0} \frac{1}{h}\left[(\lambda + h - A)^{-1} - (\lambda - A)^{-1}\right] \quad \textit{(h complex)}$$

exists in operator-norm for all $\lambda \in \rho(A)$.)

Note. The second formula is fairly easy to derive from the resolvent identity. To prove the first formula you simply have to sit down and estimate (uniform convergence on each finite interval, and everything goes to zero exponentially at ∞).

A useful tool which is used in this proof is the following.

Definition 2.8.3 We call

$$A_\lambda := \lambda A(\lambda - A)^{-1} = \lambda(\lambda(\lambda - A)^{-1} - 1) = A J_\lambda$$

the **Yoshida approximation** of A.

Lemma 2.8.4 *$A_\lambda z \to Az$ as $\alpha \to \infty$ for all $z \in \mathscr{D}(A)$.*

Corollary 2.8.5 *$A : Z \supset \mathscr{D}(A) \mapsto Z$ is the generator of a C_0-semigroup T on Z satisfying $\|T^t\| \le M e^{\omega t}$, $t \ge 0$, if and only if the following conditions hold:*

(i) *$\mathscr{D}(A)$ is dense in Z.*

(ii) *Every real $\lambda > \omega$ belongs to $\rho(A)$, and for all $\lambda > \omega$ and $n = 0, 1, 2, 3, \dots$*

$$\left\| \left(\frac{\partial}{\partial \lambda}\right)^n (\lambda - A)^{-1} \right\| \le \frac{Mn!}{(\lambda - \omega)^{n+1}}.$$

Corollary 2.8.6 *A is the generator of a C_0-semigroup satisfying (take $M \approx 1$)*

$$\|T^t\| \le e^{\omega t}$$

if and only if

(i) *$\mathscr{D}(A)$ is dense in Z and*

(ii) *every real $\lambda > \omega$ belongs to $\rho(A)$ and*

$$\|(\lambda - A)^{-1}\| \leq \frac{1}{\lambda - \omega},\ \lambda > \omega.$$

PROOF. This follows from Theorem 2.8.1 since

$$\begin{aligned}
\|(\lambda - A)^{-n}\| &= \|(\lambda - A)^{-1}(\lambda - A)^{-1} \dots (\lambda - A)^{-1}\| \\
&\leq \|(\lambda - A)^{-1}\|\|(\lambda - A)^{-1}\| \dots \|(\lambda - A)^{-1}\| \\
&= \|(\lambda - A)^{-1}\|^n \leq \frac{1}{(\lambda - \omega)^n}.
\end{aligned}$$

Example 2.8.7 (2.1.13) The generator of the semigroup in Theorem 2.5.6 is given by:

$$\mathscr{D}(A) = \left\{ z \in Z \;\middle|\; \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle z, \phi_n \rangle|^2 < \infty \right\},$$

and

$$Az = \sum_{n=1}^{\infty} \lambda_n \langle z, \phi_n \rangle \phi_n.$$

PROOF. Homework (maybe). The constants λ_n need not be real, but the must satisfy

$$\sup_n \Re \lambda_n < \infty.$$

2.9 Contraction Semigroups

Definition 2.9.1 T is a **contraction** semigroup if $\|T^t\| \leq 1$ for all $t > 0$.

Corollary 2.9.2 *A is the generator of a contraction semigroup if and only if the condition in Corollary 2.8.6 holds with $\omega = 0$:*

(i) *$\mathscr{D}(A)$ is dense in Z and*

(ii) *every real $\lambda > 0$ belongs to $\rho(A)$ and*

$$\|\lambda(\lambda - A)^{-1}\| \leq 1,\ \lambda > 0.$$

(The norm of the "approximate identity" J_λ is ≤ 1.)

There is another characterization of a contraction semigroup which uses the **adjoint** A^* of A.

Definition 2.9.3 Let $A : \mathscr{D}(A) \mapsto Z$ be a linear operator, with $\mathscr{D}(A)$ dense in Z. Then we define another operator A^* as follows:

(i) $\mathscr{D}(A^*)$ consists of all those $z \in Z$ for which there is a finite constant M such that

$$\|\langle Ax, z\rangle\|_Z \leq Mx,$$

for all $x \in \mathscr{D}(A)$.

(ii) We define A^*z for $z \in \mathscr{D}(A^*)$ as follows: If $z \in \mathscr{D}(A^*)$, then the linear map $x \mapsto \langle Ax, z\rangle$ can be extended to a bounded linear map $Z \mapsto \mathbb{C}$ (a bounded functional). By the "Riesz' representation theorem", each such functional can be written in the form $\langle x, y\rangle$ for some unique $y \in Z$. We can therefore define $A^*z = y$, $z \in \mathscr{D}(A^*)$.

Another way to put this:

Lemma 2.9.4 *Let $A : \mathscr{D}(A) \mapsto Z$ be a linear operator with $\mathscr{D}(A)$ dense in Z. Then*

$$\langle Ax, z\rangle = \langle x, A^*z\rangle \tag{2.10}$$

for all $x \in \mathscr{D}(A)$ and $z \in \mathscr{D}(A^*)$, and the preceding identity determines A^* uniquely on $\mathscr{D}(A^*)$.

Proof. Equation (2.10) is true, because we have

$$\langle Ax, z\rangle = \langle x, y\rangle$$

where $y = A^*z$.

Uniqueness: Suppose that both B and C maps $\mathscr{D}(A^*)$ into Z, and that

$$\langle Ax, z\rangle = \langle x, Bz\rangle = \langle x, Cz\rangle$$

for all $x \in \mathscr{D}(A)$ and all $z \in \mathscr{D}(A^*)$. Then $\langle x, (B-C)z\rangle = 0$ for all $x \in \mathscr{D}(A)$, $z \in \mathscr{D}(A^*)$. Fix z. If $x \in Z$ is arbitrary, then we can find $x_n \in \mathscr{D}(A)$, $x_n \to x$ as $n \to \infty$ (since $\mathscr{D}(A)$ is dense). Then

$$\langle x, (B-C)z\rangle = \lim_{n\to\infty} \langle x_n, (B-C)z\rangle = \lim_{n\to\infty} 0 = 0.$$

Thus, $\langle x, (B-C)x \rangle = 0$ for all $x \in \mathscr{D}(A),\ z \in \mathscr{D}(A^*)$. Take $x = (B-C)z$ to get

$$\begin{aligned} &\|(B-C)z\| = 0 \text{ for all } z \in \mathscr{D}(A^*) \\ &\Rightarrow Bz = Cz \text{ for all } z \in \mathscr{D}(A^*). \end{aligned}$$

□

Note. It is important that $\overline{\mathscr{D}(A)}$ is dense in Z. Otherwise A^* is not defined.

Note 2.9.5 The adjoint operator A^* is always closed. If A is closed, then $\mathscr{D}(A^*)$ is dense in Z, and $(A^*)^* = A$ (with $\mathscr{D}((A^*)^*) = \mathscr{D}(A)$).

PROOF. See [Rud73], Theorem 13.12.

Warning. Note 2.9.5 is true in a Hilbert space, but not in every Banach space (only in the reflexive ones).

Lemma 2.9.6 $(\alpha - A)^* = \overline{\alpha} - A^*$.

PROOF. Easy.

Theorem 2.9.7 *An operator $A : Z \supset \mathscr{D}(A) \mapsto Z$ is the generator of a C_0 contraction semigroup if and only if the following conditions hold:*

(i) *A is closed and $\mathscr{D}(A)$ is dense in Z.*

(ii) *$\|(\lambda - A)x\| \geq \lambda\|x\|$ for all $\lambda > 0,\ x \in \mathscr{D}(A)$.*

(iii) *$\|(\lambda - A^*)z\| \geq \lambda\|z\|$ for all $\lambda > 0,\ z \in \mathscr{D}(A^*)$.*

Alternatively we may replace (iii) by

(iii)' *Every real $\lambda > 0$ belongs to $\rho(A)$.*

Definition 2.9.8 An operator A satisfying (ii) above is called **dissipative**. An operator satisfying (ii) and (iii)' is called **maximal dissipative** (it is impossible to extend the definition of A to a larger domain without destroying the dissipativity).

Corollary 2.9.9 *A is the generator of a contraction semigroup if and only if A is maximal dissipative.*

PROOF of Theorem 2.9.7. Assume first that (i), (ii) and (iii)' hold. Then (iii)' implies that $(\lambda - A)$ is invertible. Let $x \in \mathscr{D}(A)$ and put $y = (\lambda - A)x \iff x = (\lambda - A)^{-1}y$. By (ii),

$$\|x\| = \|(\lambda - A)^{-1}y\| \leq \frac{1}{\lambda}\|y\|.$$

This is true for all $y \in Z$ since $\lambda - A$ maps $\mathscr{D}(A)$ onto Z. By Corollary 2.9.2 A generates a contraction semigroup.

Assume next that (i), (ii) and (iii) hold. We claim that this implies (iii)'. Clearly, by (ii), $\lambda - A$ is one-to-one. We claim that range(λ – A) (the range of $\lambda - A$) is closed in Z: If $z_n \in$ range(λ – A) (i.e., $z_n = (\lambda - A)x_n$ for some $x_n \in \mathscr{D}(A)$), and if $z_n \to z$ then $\{z_n\}$ is a Cauchy sequence. Then by (ii), also $\{x_n\}$ is a Cauchy sequence and hence it converges to some $x \in Z$. Then $x_n \to x$ in Z and $(\lambda - A)x_n \to z$ in Z, and since $\lambda - A$ is closed, we must have $x \in \mathscr{D}(A)$ and $z = (\lambda - A)z$. Thus $x \in$ range(λ – A), so range(λ – A) is indeed closed.

Next we claim that range(λ – A) is also dense in Z (if it is both closed and dense, then it is the whole Z). If not, then there is some $z \in Z$, $z \neq 0$, such that $z \perp (\lambda - A)x$ for all $x \in \mathscr{D}(A)$. Thus

$$\langle z, (\lambda - A)x \rangle = 0 \text{ for all } x \in \mathscr{D}(A).$$

This implies that $z \in \mathscr{D}(A^*)$ and that

$$\langle (\overline{\lambda} - A^*)z, x \rangle = \langle z, (\lambda - A)x \rangle = 0 = \langle 0, x \rangle.$$

By Lemma 2.9.4, $(\overline{\lambda} - A^*)z = 0$. Use this in (iii) to get $\|(\overline{\lambda} - A^*)z\| \geq \lambda\|z\|$ (λ is real). But we assumed that $z \neq 0$. Therefore no such z exists, i.e., range(λ – A) is dense. Now we have showed that (i), (ii) and (iii) implies (i), (ii) and (iii)', so A generates a contraction semigroup in this case too.

Converse direction: If A generates a contraction semigroup, then (i), (ii) and (iii)' hold (see Corollary 2.9.2). Condition (iii) remains. Proved as follows:

Proof #1. We shall see later that also $(T^t)^*$ is a semigroup whose generator is A^*. $\|(T^t)^*\| = \|T^t\|$, so $(T^t)^*$ is a contraction semigroup if and only if T^t is a contraction semigroup. If we replace A by A^* then (ii) becomes (iii). Thus (iii) holds.

Proof #2. Take $x \in \mathscr{D}(A^*)$. Then

$$\begin{aligned}
\|(\lambda - A^*)x\| &= \sup_{\|z\|=1} |\langle(\lambda - A^*)x, z\rangle| \quad (\mathscr{D}(A) \text{ is dense}) \\
&= \sup_{\substack{\|z\|=1 \\ z\in\mathscr{D}(A)}} |\langle(\lambda - A^*)x, z\rangle| \\
&= \sup_{\substack{\|z\|=1 \\ z\in\mathscr{D}(A)}} |\langle x, (\lambda - A)z\rangle| \qquad (\text{take } z = (\lambda - A)^{-1}y) \\
&= \sup_{\|(\lambda-A)^{-1}y\|=1} |\langle x, y\rangle| \qquad (\text{use (ii), take } y = \lambda x/\|x\|) \\
&\geq \sup_{\|y\|\leq\lambda} |\langle x, y\rangle| = \lambda\|x\|.
\end{aligned}$$

Thus, $\|(\lambda - A^*)x\| \geq \lambda\|x\|$ for $x \in \mathscr{D}(A^*),\ \lambda > 0$.

2.10 Dual Semigroups

Theorem 2.10.1 *The following conditions are equivalent.*

(i) *T^t is a C_0-semigroup with generator A,*

(ii) *$(T^t)^*$ is a C_0-semigroup with generator A^*.*

Proof. (Outline.) Easy to show that if T^t is a semigroup then $(T^t)^*$ is a semigroup. If T^t is strongly continuous then $(T^t)^*$ is strongly continuous, this is hard to show. See [CZ95] or [Sta05]. (The most difficult part is $\mathscr{D}(A^*)$ is dense in Z.)

Chapter 3

The Cauchy Problem

3.1 The Homogeneous Cauchy Problem

Recall from Theorem 2.6.4 that the **homogeneous Cauchy problem**

$$\begin{cases} \dot{z}(t) = Az(t), \quad t \geq 0, \\ z(0) = z_0. \end{cases} \tag{3.1}$$

(also called "initial value problem") has at least one continuously differentiable solution $z(t)$ provided

(i) A is the generator of a C_0-semigroup,

(ii) $z_0 \in \mathscr{D}(A)$,

namely the solution $z(t) = T^t z_0$. This solution satisfies:

(A) $z(t) \in \mathscr{D}(A)$ for all $t \geq 0$,

(B) z is continuously differentiable on $[0, \infty)$,

(C) equation (3.1) holds for all $t \in [0, \infty)$.

Can there exist more than one solution? Answer: No (follows from Corollary 3.2.4 below).

3.2 The Inhomogeneous Cauchy Problem

We study a more general problem at the same time.

$$\begin{cases} \dot{z}(t) = Az(t) + f(t), \quad t \geq 0, \\ z(0) = z_0, \end{cases} \tag{3.2}$$

where $f \in C(\mathbb{R}^+; Z)$. Sometimes $f(t)$ is defined only for $t \in [0, \tau]$, and we restrict t in (3.2) to belong to $[0, \tau]$ also ($\tau > 0$ is given).

Definition 3.2.1 We call z a **classical** solution of (3.2) on the interval $[0, \tau]$ if

(i) $f \in C([0, \tau]; Z)$

(ii) $z \in C^1([0, \tau]; Z)$

(iii) $z(t) \in \mathscr{D}(A)$ for all $t \in [0, \tau]$

(iv) (3.2) holds on $[0, \tau]$, i.e., $\dot{z}(t) = Az(t) + f(t)$ for all $t \in [0, \tau]$ and $z(0) = z_0$.

Lemma 3.2.2 *(3.1.2) Let A be the generator of a C_0-semigroup T. If (3.2) has a classical solution z on some interval $[0, \tau]$, then it must be given by*

$$z(t) = T^t z_0 + \int_0^t T^{t-s} f(s)\, ds.$$

Proof. We claim that the function $s \mapsto T^{t-s} z(s)$ is continuously differentiable. This is shown as follows:

$$\begin{aligned} &\frac{1}{h}[T^{t-s-h} z(s+h) - T^{t-s} z(s)] \\ &= \underbrace{T^{t-s-h} \frac{z(s+h) - z(s)}{h}}_{(1)} + \underbrace{\frac{1}{h}[T^{t-s-h} - T^{t-s}] z(s)}_{(2)}. \end{aligned}$$

As $h \to 0$, $\frac{z(s+h)-z(s)}{h} \to \dot{z}(s)$, and T^{t-s-h} tends strongly to T^{t-s}, so the product tends to $T^{t-s}\dot{z}(s)$. Thus, (1) $\to T^{t-s}\dot{z}(s)$ as $h \to \infty$. The term (2) is a little more complicated. If $h < 0$, then

$$(2) = -T^{t-s} \frac{1}{-h}[T^{-h} - 1] z(s) \to -T^{t-s} A z(s)$$

as $h \to 0-$ since $z(s) \in \mathscr{D}(A)$. If $h > 0$, then

$$(2) = \underbrace{T^{t-s-h}}_{\to T^{t-s} \text{ strongly}} \underbrace{\frac{1}{h}[1 - T^h]z(s)}_{\to -Az(s)} \to -T^{t-s}Az(s),$$

so the whole limit exists, and

$$\frac{d}{ds}[T^{t-s}z(s)] = T^{t-s}[\dot{z}(s) - Az(s)] = T^{t-s}f(s)$$

by (3.2). This is a continuous function. Integrating over $[0, t]$ we get

$$\begin{aligned} \int_0^t T^{t-s}f(s)\,ds &= \left[T^{t-s}z(s)\right]_0^t \\ &= z(t) - T^t z(0) \qquad \text{(use (3.2))} \\ &= z(t) - T^t z_0. \end{aligned}$$

□

Corollary 3.2.3 *If A is the generator of a C_0-semigroup T, and if $f \in C([0, \tau]; Z)$, then (3.2) has at most one classical solution on $[0, \tau]$ (namely the one in Lemma 3.2.2).*

Corollary 3.2.4 *If A is the generator of a C_0-semigroup, then the homogeneous equation $\dot{z}(t) = Az(t)$, $t \geq 0$, $z(0) = z_0$ has a classical solution if and only if $z_0 \in \mathscr{D}(A)$. This solution is unique.*

PROOF. If $z_0 \in \mathscr{D}(A)$, then $T^t z_0$ is a classical solution, and it is unique by Corollary 3.2.3. If z is a classical solution, then z is differentiable at zero, so $z(0) = z_0 \in \mathscr{D}(A)$ (and $Az_0 = \dot{z}(0)$). □

Thus, *equation (3.2) does not always have a classical solution.* (Take $f \equiv 0$, $z_0 \notin \mathscr{D}(A)$.) However, the following theorem is true (this theorem is not found in [CZ95]).

Theorem 3.2.5 *Let A be the generator of a C_0-semigroup T, let $z_0 \in Z$, $f \in C([0, \tau]; Z)$, and define*

$$z(t) = T^t z_0 + \int_0^t T^{t-s}f(s)\,ds. \tag{3.3}$$

Then z is a classical solution of (3.2) if and only if $z \in C^1([0, \tau]; Z)$. (In particular, this implies $z_0 \in \mathscr{D}(A)$.)

PROOF. We know from Lemma 3.2.2 that if z is a classical solution then (3.3) holds, and every classical solution must belong to $C^1([0,\tau];Z)$.

Conversely, suppose that z given by (3.3) belongs to $C^1([0,\tau];Z)$. The equation (3.3) implies that $z(0)=z_0$. By assumption, the following limit exists.

$$\begin{aligned}
\dot{z}(0) &= \lim_{h\to 0+}\frac{1}{h}(z(h)-z_0)\\
&= \lim_{h\to 0}\left[\frac{1}{h}(T^h-1)z_0+\underbrace{\frac{1}{h}\int_0^h T^{h-s}f(s)\,ds}_{\to f(0)}\right]\\
&= f(0)+\lim_{h\to 0}\frac{1}{h}(T^h-1)z_0.
\end{aligned}$$

Therefore $z_0\in\mathscr{D}(A)$ and $\dot{z}(0)=f(0)+Az_0$. Now define $z=u+v$, where

$$\begin{cases} u(t)=T^t z_0, & t\geq 0,\\ v(t)=\int_0^t T^{t-s}f(s)\,ds & t\geq 0.\end{cases}$$

Then u is a classical solution of

$$\begin{cases} \dot{u}(t)=Au(t), & t\geq 0,\\ u(0)=z_0.\end{cases}$$

So if we can show that v is a classical solution of the equation

$$\begin{cases} \dot{v}(t)=Av(t)+f(t) & t\geq 0\\ v(0)=0,\end{cases} \tag{3.4}$$

then their sum $z=u+v$ is a classical solution of (3.2). Thus, we got rid of z_0. We must still show that v is a classical solution of (3.4). We know that $v\in C^1([0,\tau];Z)$. On one hand

$$v(t+h)-v(t)=\int_0^{t+h}T^{t+h-s}f(s)\,ds-\int_0^t T^{t-s}f(s)\,ds.$$

On the other

$$(T^h-1)v(t)=\int_0^t (T^{t+h-s}-T^{t-s})f(s)\,ds.$$

Compare these two to get

$$\begin{aligned}
\frac{1}{h}(T^h-1)v(t) &= \frac{v(t+h)-v(t)}{h}-\frac{1}{h}\int_t^{t+h}T^{t+h-s}f(s)\,ds\\
&= \underbrace{\frac{1}{h}[v(t+h)-v(t)]}_{\to v'(t)}-\underbrace{\frac{1}{h}\int_0^h T^s f(t+h-s)\,ds}_{\to f(t)} \qquad (3.5)
\end{aligned}$$

Thus, $v(t) \in \mathscr{D}(A)$, and

$$Av(t) = v'(t) - f(t)$$

for $0 \leq t \leq \tau$. Furthermore $v(0) = 0$. This means that v is a classical solution of (3.4). $\square$

Theorem 3.2.6 *(3.1.3) If A generates a C_0-semigroup T, if $z_0 \in \mathscr{D}(A)$ and if $f \in C^1([0,\tau]; Z)$, then $z(t)$ defined in (3.3) is a classical solution to (3.2).*

PROOF. It suffices to show that z is continuously differentiable (see Theorem 3.2.5). We know that $u(t) = T^t z_0 \in C^1([0,\tau]; Z)$. Must show that $v(t) = \int_0^t T^{t-s} f(s)\, ds \in C^1$. Compute (assume $h > 0$)

$$\frac{1}{h}[v(t+h) - v(t)] = \frac{1}{h}\int_0^{t+h} T^{t+h-s} f(s)\, ds - \frac{1}{h}\int_0^t T^{t-s} f(s)\, ds$$

(change integration variable)

$$= \frac{1}{h}\int_0^t T^{t-s}[f(s+h) - f(s)]\, ds + \frac{1}{h}\int_0^h T^{t+h-s} f(s)\, ds$$

($\frac{1}{h}[f(s+h) - f(s)] \to f'(s)$ uniformly on $[0,\tau]$ as $h \to 0+$ since f' continuous)

$$\to \int_0^t T^{t-s} f'(s)\, ds + T^t f(0)$$

as $h \to 0+$.

A similar computation can be carried out when $h \to 0-$.

Therefore $v \in C^1([0,\tau]; Z)$ and

$$\dot{v}(t) = T^t f(0) + \int_0^t T^{t-s} f'(s)\, ds.$$

$\square$

Another way to check if (3.3) is a classical solution of (3.2) is the following:

Theorem 3.2.7 *Let A be the generator of a C_0-semigroup T, let $z_0 \in Z$, $f \in C([0,\tau]; Z)$ and define z by (3.3). Then z is a classical solution of (3.2) if and only if $z(t) \in \mathscr{D}(A)$ for all $t \in [0,\tau]$ and $Az \in C([0,\tau]; Z)$.*

PROOF. Necessity is obvious: By the definition of a classical solution (see Definition 3.2.1) $z(t) \in \mathscr{D}(A)$ for all $t \in [0,\tau]$ and $Az(t) = \dot{z}(t) - f(t) \in C([0,\tau];Z)$.

Conversely, suppose that $z(t) \in \mathscr{D}(A)$ for all $t \in [0,\tau]$ and that Az is continuous. Then $z_0 \in \mathscr{D}(A)$ and we can get rid of the term $T^t z_0$ in the same way as in the proof of Theorem 3.2.5. This leaves the function

$$v(t) = \int_0^t T^{t-s} f(s)\, ds.$$

By (3.5),

$$\frac{v(t+h) - v(t)}{h} = \frac{1}{h}(T^h - 1)v(t) + \frac{1}{h}\int_0^h T^s f(t+h-s)\, ds.$$

Since $v(t) \in \mathscr{D}(A)$, the limit of the right-hand side exists as $h \to 0+$, so v has a right-derivative:

$$\lim_{h\to 0+} \frac{v(t+h) - v(t)}{h} = Av(t) + f(t) :$$

A similar computation which uses the identity (valid for $0 \le h \le t$)

$$\frac{v(t-h) - v(t)}{-h} = -\frac{1}{h}(1 - T^h)v(t-h) + \frac{1}{h}\int_0^h T^s f(t-s)\, ds$$

(and the continuity of Av) shows that v also has a left-derivative, and that

$$\dot{v}(t) = Av(t) + f(t).$$

The right-hand side is continuous, so $v \in C^1([0,\tau];Z)$, and by Theorem 3.2.5, v is a classical solution of (3.2). □

The following result follows from Theorem 3.2.7.

Theorem 3.2.8 *If A generates a C_0-semigroup T, $z_0 \in \mathscr{D}(A)$, $f \in C([0,\tau];X)$, $f(t) \in \mathscr{D}(A)$ for all $t \in [0,\tau]$ and $Af(t)$ is continuous with values in Z, then the function z given by (3.3) is again a classical solution to (3.2).*

PROOF. Homework.

Definition 3.2.9 We call the function (3.3)

$$z(t) = T^t z_0 + \int_0^t T^{t-s} f(s)\, ds$$

the **mild solution** (mild, mieto) of

$$\begin{cases} \dot{z}(t) = Az(t) + f(t) & t \geq 0 \\ z(0) = z_0 \end{cases}$$

(often also called **weak** (svag, hiekko)). The equation (3.3) above is called the *variation of constants formula.*

3.3 A Bounded Feedback Perturbation

If we in the equation

$$\begin{cases} \dot{z}(t) = Az(t) + f(t) & t \geq 0, \\ z(0) = z_0 \end{cases}$$

make a "state feedback connection", i.e., we let f depend on z through a formula of the type

$$f(t) = Dz(t), \ t \geq 0,$$

where $D \in \mathcal{L}(Z)$, then the "closed loop system" where f has been replaced by $Dz(t)$ is of the form

$$\begin{cases} \dot{z}(t) = (A + D)z(t) & t \geq 0, \\ z(0) = z_0. \end{cases}$$

This system has a unique mild solution if $(A+D)$ generates a C_0-semigroup. Replacing f by Dz in the variation of constants formula (3.3) we get

$$z(t) = T^t z_0 + \int_0^t T^{t-s} Dz(s) \, ds, \ t \geq 0. \tag{3.6}$$

Theorem 3.3.1 *(3.2.1) If A generates a C_0-semigroup T, and if $D \in \mathcal{L}(H)$, then $A + D$ is the generator of a C_0-semigroup S. This semigroup is the unique strongly continuous solution of the equation*

$$S^t z_0 = T^t z_0 + \int_0^t T^{t-s} D S^s z_0 \, ds, \tag{3.7}$$

and it also satisfies

$$S^t z_0 = T^t z_0 + \int_0^t S^{t-s} D T^s z_0 \, ds. \tag{3.8}$$

Moreover, if $\|T^t\| \leq M e^{\omega t}$ for $t \geq 0$, then

$$\|S^t\| \leq M e^{(\omega + M\|D\|)t}, \ t \geq 0.$$

Proof. (Only outline.)

Step 1. Use the "contraction mapping principle" (the Banach fixed point theorem) to show that the equation (3.6) has a unique continuous solution z for every $z_0 \in Z$. Define $S^t z_0 = z(t)$, $t \geq 0$.

Step 2. Show that S is a C_0-semigroup.

Step 3. Show that the generator of S is $A + D$.

Step 4. Show that also (3.8) holds.

Note: $A+D$ has the same domain as A. The limit $\lim_{h\to 0+} \frac{1}{h}(S^h - 1)z_0$ exists if and only if $\lim_{h\to 0+} \frac{1}{h}(T^h - 1)z_0$ exists and

$$A + D = \lim_{h\to 0+} \frac{1}{h}(S^h - 1)z_0 = \lim_{h\to 0+} \frac{1}{h}(T^h - 1)z_0 + D.$$

Chapter 4

Controllability and Observability

4.1 The system $\Sigma = (A, B, C, D)$

We now go gradually back to the full system Σ:

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), & t \geq 0, \\ y(t) = Cz(t) + Du(t), & t \geq 0, \\ z(0) = z_0. \end{cases} \tag{4.1}$$

Here we assume that

$$\begin{array}{lcl} A & & \text{generates a } C_0\text{-semigroup } T, \\ B & \in & \mathcal{L}(U; Z), \\ C & \in & \mathcal{L}(Z; Y), \\ D & \in & \mathcal{L}(U; Y). \end{array}$$

Definition 4.1.1 Let $z_0 \in Z$ and $u \in L^2_{\text{loc}}(\mathbb{R}^+; U)$. By **the solution** of (4.1) we mean the mild solution, given by

$$\begin{aligned} z(t) &= T^t z_0 + \int_0^t T^{t-s} Bu(s)\, ds,\ t \geq 0 \\ y &= Cz(t) + Du(t). \end{aligned} \tag{4.2}$$

(Thus, $z \in C(\mathbb{R}^+; Z)$ and $y \in L^2_{\text{loc}}(\mathbb{R}^+; Y)$.)

Note. The integral in (4.2) can be interpreted as a classical Riemann integral if u is continuous, and in this case also y is continuous. In general we must interpret this as a Lebesgue integral (or alternatively, replace u by a sequence $u_n \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^+; U)$ where each u_n is continuous, replace u by u_n in (4.2) and let $n \to \infty$).

4.2 Controllability in Time $\tau < \infty$

To begin with we ignore the output part of the system and the initial state and only look at the

$$\boxed{\text{interplay between input and state.}}$$

Therefore, it is enough to look at the equation

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), & t \geq 0, \\ z(0) = 0. \end{cases} \tag{4.3}$$

Definition 4.2.1 The **controllability or reachability map** $\mathcal{B}^\tau$ over the time interval $[0, \tau]$ is the operator

$$\mathcal{B}^\tau u = \int_0^\tau T^{\tau - s} Bu(s)\, ds.$$

Note that $\mathcal{B}^\tau$ is the mapping u to $z(\tau)$, where z is the solution of (4.3). It is a bounded linear operator from $L^2([0, \tau]; U)$ to Z. This is the controllability map used throughout in [CZ95].

Later we shall allow the 'final time' τ to vary, and the formulas become slightly simpler if we shift the starting time to $-\tau$ (and the 'final time' to be zero):

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), & t \geq -\tau, \\ z(-\tau) = 0. \end{cases} \tag{4.4}$$

Definition 4.2.2 The **controllability or reachability map** $\mathcal{B}_\tau$ over the time interval $[-\tau, 0]$ is the operator

$$\mathcal{B}_\tau u = \int_{-\tau}^0 T^{-s} Bu(s)\, ds.$$

Note that $\mathcal{B}_\tau$ is the mapping u to $z(0)$, where z is the solution of (4.4). It is a bounded linear operator from $L^2([-\tau,0];U)$ to Z. It has the same range as $\mathcal{B}^\tau$.

In all the results of this subsection and the next we can as well replace $\mathcal{B}_\tau$ by $\mathcal{B}^\tau$, if we at the same time replace $L^2([-\tau,0];U)$ by $L^2([0,\tau];U)$. (Some of the formulas become slightly more complicated. This is what [CZ95] does.)

Definition 4.2.3 **(a)** The system $\Sigma = (A, B, -)$ is **exactly** (exakt, tarkasti) **controllable** (styrbar, reglerbar, säädettävä, ohjattava) **in time** τ if the range of $\mathcal{B}_\tau$ is the whole space Z.

(b) The same system is **approximatively controllable in time** τ if the range of $\mathcal{B}_\tau$ is dense in Z.

(c) The **controllability Gramian** (Gramoperator) is the operator

$$L_\mathcal{B}^\tau = \mathcal{B}_\tau \mathcal{B}_\tau^*.$$

What is $\mathcal{B}_\tau^*$?

For each bounded linear operator $A \in \mathcal{L}(Z_1; Z_2)$ we define the adjoint in the same way as the unbounded adjoint in Definition 2.9.3. If $A \in \mathcal{L}(Z_1; Z_2)$, then $\mathscr{D}(A) = Z_1$, and the adjoint operator is also bounded. It is still characterized by the same identity,

$$\langle Ax, z\rangle_{Z_2} = \langle x, A^* z\rangle_{Z_1}, \ x \in Z_1, \ z \in Z_2.$$

(Note that if $A : Z_1 \mapsto Z_2$, then $A^* : Z_2 \mapsto Z_1$; the first inner product is taken in Z_2 and the second in Z_1.) Thus,

$$\boxed{\begin{array}{c} A \in \mathcal{L}(Z_1; Z_2), \ A^* \in \mathcal{L}(Z_2; Z_1), \text{ and} \\ \langle Ax, z\rangle_{Z_2} = \langle x, A^* z\rangle_{Z_1} \text{ for all } x \in Z_1, \ z \in Z_2. \end{array}}$$

Back to $\mathcal{B}_\tau^*$. $\mathcal{B}_\tau$ maps $L^2([-\tau,0];U)$ into Z, so $\mathcal{B}_\tau^*$ maps Z into $L^2([-\tau,0];U)$. For any $z \in Z$ and $u \in L^2([-\tau,0];U)$,

$$\begin{aligned}
\langle u, \mathcal{B}_\tau^* z\rangle_{L^2([-\tau,0];U)} &= \langle \mathcal{B}_\tau u, z\rangle_Z \\
&= \left\langle \int_{-\tau}^0 T^{-s} B u(s)\, ds, z \right\rangle_Z \\
&= \int_{-\tau}^0 \langle T^{-s} B u(s), z\rangle_Z \, ds \\
&= \int_{-\tau}^0 \langle u(s), B^* (T^{-s})^* z\rangle_U \, ds.
\end{aligned}$$

Since the function $\mathcal{B}_\tau^* z$ has the same inner product in $L^2([-\tau,0];U)$ with every $u \in L^2([-\tau,0];U)$ as the function $s \mapsto B^*(T^{-s})^* z$, this means that

$$(\mathcal{B}_\tau^* z)(s) = B^*(T^{-s})^* z, \ -\tau \le s \le 0.$$

To get the *controllability Gramian* we compute

$$\begin{aligned} L_\mathcal{B}^\tau z &= \mathcal{B}_\tau \mathcal{B}_\tau^* z \\ &= \mathcal{B}_\tau u \qquad \text{(where } u(s) = B^*(T^{-s})^* z) \\ &= \int_{-\tau}^0 T^{-s} B B^* (T^{-s})^* z \, ds \qquad (-s \to s) \\ &= \int_0^\tau T^s B B^* (T^s)^* z \, ds. \end{aligned}$$

Thus, we have proved the following.

Lemma 4.2.4

$$L_\mathcal{B}^\tau z = \int_0^\tau T^s B B^* T^{*s} z \, ds,$$

where we have denoted $T^{*s} = (T^s)^* =$ the dual semigroup evaluated at s.

Note 4.2.5 *A similar computation shows that we also have $L_\mathcal{B}^\tau = \mathcal{B}^\tau(\mathcal{B}^\tau)^*$ (where we use the controllability map over the time interval $[0,\tau]$ instead.*

Lemma 4.2.6 *$L_\mathcal{B}^\tau$ is self-adjoint and positive, i.e., $(L_\mathcal{B}^\tau)^* = L_\mathcal{B}^\tau$, and*

$$\langle L_\mathcal{B}^\tau z, z\rangle \ge 0, \ z \in Z.$$

Proof. Self-adjoint because

$$(L_\mathcal{B}^\tau)^* = (\mathcal{B}_\tau \mathcal{B}_\tau^*)^* = (\mathcal{B}_\tau^*)^* \mathcal{B}_\tau^* = \mathcal{B}_\tau \mathcal{B}_\tau^*.$$

(Every bounded operator B satisfies $(B^*)^* = B$.) Positive because

$$\langle L_\mathcal{B}^\tau z, z\rangle = \langle \mathcal{B}_\tau \mathcal{B}_\tau^* z, z\rangle = \langle \mathcal{B}_\tau^* z, \mathcal{B}_\tau^* z\rangle = \|\mathcal{B}_\tau^* z\|^2 \ge 0.$$

$\square$.

Back to Definition 4.2.3. *Why do we need two different controllability notions?*

Answer: In the finite dimensional case we do not, but the infinite dimensional case is problematic.

Theorem 4.2.7 *(4.1.5) If U is finite dimensional and Z infinite dimensional, then $\Sigma(A,B,-)$ is not exactly controllable in time τ for any $\tau > 0$.*

Idea of PROOF. The operator $\mathcal{B}_\tau$ is compact in this case (it maps every bounded set in $L^2([0,\tau];U)$ into a totally bounded subset of Z), and the range of a compact operator is never the whole space (if it is infinite dimensional). (Use Arzela-Ascoli.)

Corollary 4.2.8 *Both $\mathcal{B}_\tau$ and $L_{\mathcal{B}}^\tau$ are compact if U is finite dimensional.*

Shown as a part of the proof of Theorem 4.2.7.

Thus, if you need exact controllability, then you must either

- use ∞-dimensional U, or
- use an unbounded operator B.

(Here "unbounded" means that B does not map U into Z but into some "larger space" $V \supset Z$. For example, $Z = L^2$, and V consists of distributions.)

4.3 Equivalent Controllability Conditions

$\boxed{\text{Controllability is preserved under feedback.}}$

Lemma 4.3.1 *(4.1.6) The following conditions are equivalent:*

(i) *$\Sigma(A,B,-)$ is exactly controllable in time τ,*

(ii) *$\Sigma(A+\alpha,B,-)$ is exactly controllable in time τ for some $\alpha \in \mathbb{C}$,*

(iii) *$\Sigma(A+\alpha,B,-)$ is exactly controllable in time τ for all $\alpha \in \mathbb{C}$,*

(iv) *$\Sigma(A+BF,B,-)$ is exactly controllable in time τ for some $F \in \mathcal{L}(Z;U)$,*

(v) *$\Sigma(A+BF,B,-)$ is exactly controllable in time τ for all $F \in \mathcal{L}(Z;U)$.*

These conditions are also equivalent if we throughout replace "exactly controllable" by "approximately controllable".

PROOF. Fairly simple. Based on the fact that all the different controllability maps listed above have the same ranges. See [CZ95]. It is not formulated in exactly this way in [CZ95], but the proof remains the same.

The following theorem connects the different controllability properties to the corresponding properties of the Gramian.

Theorem 4.3.2 *(4.1.7 (a)) The following conditions are equivalent:*

(i) $\langle L_{\mathcal{B}}^{\tau} z, z\rangle \geq \gamma \|z\|^2$ *for some* $\gamma > 0$*,*

(ii) $\|\mathcal{B}_\tau^* z\|^2 \geq \gamma \|z\|^2$ *for some* $\gamma > 0$*,*

(iii) $\int_0^\tau \|B^* T^{*s} z\|^2 \, ds \geq \gamma \|z\|^2$ *for some* $\gamma > 0$*,*

(iv) $\mathcal{B}_\tau^*$ *is one-to-one and has closed range,*

(v) $\Sigma(A, B, -)$ *is exactly controllable in time* τ*.*

PROOF. *(i)* $\Longleftrightarrow$ *(ii):* True, because

$$\langle L_{\mathcal{B}}^{\tau} z, z\rangle = \langle \mathcal{B}_\tau \mathcal{B}_\tau^* z, z\rangle = \|\mathcal{B}_\tau^* z\|^2.$$

(ii) $\Longleftrightarrow$ *(iii):* True because $\mathcal{B}_\tau^* z = B^*(T^{-s})^* z$, $-\tau \leq s \leq 0$, (use the same change of integration variable as on p. 44).

(ii) $\Longrightarrow$ *(iv):* Clearly, by (ii), $\mathcal{B}_\tau^*$ is one-to-one. To prove the closed range, take $y_n = \mathcal{B}_\tau^* z_n$, $y_n \to y$. This gives

$$\|y_n - y_m\| = \mathcal{B}_\tau^*(z_n - z_m),$$

so by (ii),

$$\|y_n - y_m\| \geq \gamma \|z_n - z_m\|.$$

This implies that z_n is a Cauchy-sequence, hence it converges to some $z \in Z$, and by the continuity of $\mathcal{B}_\tau^*$,

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} \mathcal{B}_\tau^* z_n = \mathcal{B}_\tau^* z.$$

Thus, y belongs to the range of $\mathcal{B}_\tau^*$.

(iv) $\Longleftrightarrow$ *(v):* $\ker(\mathcal{B}_\tau^*)^\perp = \overline{\operatorname{range}(\mathcal{B}_\tau)}$, so the range of $\mathcal{B}_\tau$ is dense in Z if and only if $\mathcal{B}_\tau^*$ is one-to-one. That $\operatorname{range}(\mathcal{B}_\tau) = \mathrm{Z}$ if and only if $\operatorname{range}(\mathcal{B}_\tau)$ is closed follows from Theorem 4.3.3 below ("the closed range theorem").

(iv) $\Longrightarrow$ *(ii):* See below. $\square$

Above we used the following.

Theorem 4.3.3 *Closed Range Theorem. For every bounded operator $A : Z_1 \mapsto Z_2$ we have*

$$\text{range}\,(\text{A}) \textit{ closed} \iff \text{range}\,(\text{A}^*) \textit{ closed.}$$

PROOF. See [Rud73], Theorem 4.14. $\square$

Theorem 4.3.4 *Closed Graph Theorem. If a linear operator A from one Hilbert space X to another Hilbert space Y is closed (has a closed graph, see p. 21) and if $\mathscr{D}(A) = X$ (the whole space), then A is bounded.*

PROOF. See [Rud73], Theorem 2.15. $\square$

PROOF of Theorem 4.3.2 continues.

(iv) $\Longrightarrow$ (ii): If $\mathcal{B}_\tau^*$ is one-to-one and has closed range, then we may regard range $(\mathcal{B}_\tau^*)$ as a Hilbert space with the same inner product as in $L^2([0,\tau];U)$ (a closed subset of a complete space is complete). Then $\mathcal{B}_\tau^*$ has an inverse defined on this space, and by the closed graph theorem $(\mathcal{B}_\tau^*)^{-1}$ is continuous. Thus, for all $z \in Z$,

$$\|z\| = \|(\mathcal{B}_\tau^*)^{-1}\mathcal{B}_\tau^* z\| \leq \|(\mathcal{B}_\tau^*)^{-1}\|\|\mathcal{B}_\tau^* z\|,$$

so (ii) holds with $\gamma = \frac{1}{\sqrt{\|(\mathcal{B}_\tau^*)^{-1}\|}}$. $\square$

There is also a similar result about *approximate controllability.*

Theorem 4.3.5 *(4.1.7 (b)) The following conditions are equivalent.*

(i) $\langle L_\mathcal{B}^\tau z, z\rangle > 0$ *for all* $z \neq 0$.

(ii) $\mathcal{B}_\tau^*$ *is one-to-one.*

(iii) *The mapping $z \mapsto B^*T^{*s}z$ from Z to $L^2([0,\tau];U)$ is one-to-one.*

(iv) $\Sigma(A, B, -)$ *is approximately controllable in time τ.*

PROOF. Similar to the proof of Theorem 4.3.2 but easier. $\square$

4.4 Observability in time τ

We now jump to another part of the system, we ignore the input and concentrate on the output instead. Thus, we study the system

$$\begin{cases} \dot{z}(t) = Az(t), & t \geq 0, \\ y(t) = Cz(t), & t \geq 0, \\ z(0) = z_0. \end{cases} \tag{4.5}$$

Question: If we know the values of $y(t)$ for all $t \in [0, \tau]$, can we then reconstruct $z(0) = z_0$?

Definition 4.4.1 (4.1.12)

(a) The **observability** map of $\Sigma(A, -, C)$ is the map $Z \mapsto L^2([0, \tau]; Y)$ given by

$$(\mathcal{C}^\tau z_0)(s) = CT^s z_0, \ 0 \leq s \leq \tau.$$

(b) This system is **approximately observable in time** τ if $\mathcal{C}^\tau$ is one-to-one, i.e., if every nonzero z_0 results in a nonzero output y.

(c) The system $\Sigma(A, -, C)$ is **exactly observable in time** τ if $\mathcal{C}^\tau$ is one-to-one and, in addition, $\mathcal{C}^\tau$ has a bounded inverse (defined on its range).

Note 4.4.2 This is also referred to as **initial observability**, i.e., the initial value z_0 can be reconstructed from the output.

Note 4.4.3 By the closed graph theorem, (c) is equivalent to the requirement that $\mathcal{C}^\tau$ is one-to-one and $\mathcal{C}^\tau$ has a closed range. (Compare this to condition (iv) in Theorem 4.3.2).

Note 4.4.4 $\mathcal{C}^\tau$ is in practice never onto, because every $y \in \operatorname{range}(\mathcal{C}^\tau)$ is continuous and not every L^2-function is continuous.

Definition 4.4.5 We call the operator

$$L_C^\tau = (\mathcal{C}^\tau)^* \mathcal{C}^\tau \in \mathcal{L}(Z)$$

the **observability gramian** of $\Sigma(A, -, C)$.

Lemma 4.4.6 *Let $\mathcal{C}^\tau$ be the observability map of $\Sigma(A,-,C)$, and let $\mathcal{B}_\tau$ be the controllability map of $\Sigma(A^*,C^*,-)$. Then*

$$\mathcal{B}_\tau^* = \Upsilon \mathcal{C}^\tau,$$

where Υ is the reflection operator

$$\Upsilon y(s) = y(-s),\ -\infty < s < \infty.$$

Note. $\Sigma(A^*,C^*,-)$ is described by the equation

$$\begin{cases} \dot{z}_d(t) = A^* z_d(t) + C^* y_d(t), & t \geq 0 \\ z_d(0) = \text{given}. \end{cases}$$

This is the dual system of the one in (4.5) (no output, but an input instead).
PROOF. We already computed $\mathcal{B}_\tau^*$ earlier,

$$\begin{aligned} \langle \mathcal{B}_\tau u, z\rangle_Z &= \left\langle \int_{-\tau}^0 (T^{-s})^* C^* u(s)\, ds, z \right\rangle \\ &= \int_{-\tau}^0 \langle (T^{-s})^* C^* u(s), z\rangle\, ds \\ &= \int_{-\tau}^0 \langle u(s), C T^{-s} z\rangle\, ds \\ &= \langle u, \Upsilon \mathcal{C}^\tau z\rangle_{L^2(0,\tau)}, \end{aligned}$$

so

$$\mathcal{B}_\tau^* = \Upsilon \mathcal{C}^\tau.$$

□

Lemma 4.4.7 *(4.1.13)*

(a) *$\Sigma(A,-,C)$ is approximately observable in time τ if and only if $\Sigma(A^*,C^*,-)$ is approximately controllable in time τ.*

(b) *$\Sigma(A,-,C)$ is exactly observable in time τ if and only if $\Sigma(A^*,C^*,-)$ is exactly controllable in time τ.*

PROOF. Use Definition 4.4.1, Lemma 4.4.6, Theorem 4.3.2 and Theorem 4.3.5. Note that $\Upsilon^*\Upsilon = I$ and that $\Upsilon^* = \Upsilon$ (it is a self-adjoint square root of the unitary operator). □

Corollary 4.4.8 *(4.1.14 (a)) The following conditions are equivalent.*

(i) $\langle L_C^\tau z, z\rangle \geq \gamma \|z\|_Z^2$ *for some* $\gamma > 0$.

(ii) $\|\mathcal{C}^\tau z\|^2_{L^2(0,\tau)} \geq \gamma \|z\|_Z^2$ *for some* $\gamma > 0$.

(iii) $\int_0^\tau \|CT^s z\|_Y^2 \, ds \geq \gamma \|z\|_Z^2$ *for some* $\gamma > 0$.

(iv) $\mathcal{C}^\tau$ *is one-to-one and has closed range.*

(v) $\Sigma(A, -, C)$ *is exactly observable in time* τ.

Proof. See Lemma 4.4.6 and Theorem 4.3.2. □

Corollary 4.4.9 *(4.1.14 (b)) The following conditions are equivalent.*

(i) $\langle L_C^\tau z, z\rangle > 0$ *for all* $z \neq 0$.

(ii) $\mathcal{C}^\tau$ *is one-to-one.*

(iii) $CT^s z = 0$ *for all* $s \in [0, \tau] \Longrightarrow z = 0$.

(iv) $\Sigma(A, -, C)$ *is approximately observable in time* τ.

Proof. See Lemma 4.4.6 and Theorem 4.3.5. □

4.5 The Reachable and Unobservable Subspaces

Example 4.5.1 (4.1.16) *Observability in infinite time.* Let T^t be the left shift on $\mathcal{W}^{1,2}(\mathbb{R}^+)$. Observation $Cz = z(0)$, $Y = \mathbb{R}$. This gives

$$(T^t z)(x) = z(x + t),\ t \geq 0,\ x \geq 0,$$

and

$$y(t) = CT^t z = z(t),\ t \geq 0.$$

Thus, $\mathcal{C}^\tau$ maps $z \in \mathcal{W}^{1,2}(\mathbb{R}^+)$ into the restriction of Z to $[0, \tau]$. In particular, $\mathcal{C}^\tau$ is not one-to-one so it is not even approximately observable. (Neither is its range closed in $L^2(0, \tau)$.) However, it is observable is we allow $\tau = \infty$!

Definition 4.5.2 **(a)** The **(approximately) reachable subset** (in infinite time) of $\Sigma(A,B,-)$ is the closure $\overline{\mathcal{R}}$ in Z of the set

$$\mathcal{R} = \bigcup_{\tau>0} \operatorname{range}(\mathcal{B}_\tau)$$

(i.e., $\mathcal{R}$ contains all states that can be reached from the zero state with some input active on some finite interval $[-\tau,0]$). We call $\Sigma(A,B,-)$ **(approximately) controllable** (in infinite time) if $\overline{\mathcal{R}} = Z$.

(b) The **nonobservable subspace** $\mathcal{N}$ (in infinite time) of $\Sigma(A,-,C)$ is

$$\mathcal{N} = \bigcap_{\tau>0} \ker(\mathcal{C}^\tau) = \{z \in Z \mid CT^t z = 0 \text{ for all } t \geq 0\}.$$

We call $\Sigma(A,-,C)$ **(approximately) observable** (in infinite time) if $\mathcal{N} = \{0\}$, i.e., if for every $0 \neq z \in Z$ there is some $t \geq 0$ such that $CT^t z \neq 0$.

Lemma 4.5.3 *(4.1.18) $\mathcal{N}$ is the largest T-invariant subspace in* $\ker(C) = \{z \in Z \mid Cz = 0\}$.

PROOF. $\mathcal{N} \subset \ker(C)$*:* Since $\mathcal{C}^\tau z = CT^t z$, $0 \leq t \leq \tau$, the condition $\mathcal{C}^\tau z = 0$ means that $CT^t z$ vanishes on $[0,\tau]$ in the L^2-sense. But $CT^t z$ is continuous, so it vanishes in the L^2-sense if and only if it is identically zero. Thus, taking $t = 0$ we get $CT^0 z = Cz = 0$, so $z \in \ker(C)$.

Invariance: Assume that $z \in \mathcal{N}$,

$$\begin{aligned}
&\iff \quad CT^s z = 0 \text{ for (almost) all } s \geq 0 \\
&\implies \quad CT^{s+t} z = 0 \text{ for (almost) all } s \geq 0 \text{ and } t \geq 0 \\
&\implies \quad CT^s(T^t z) = 0 \text{ for (almost) all } s \geq 0 \text{ and } t \geq 0
\end{aligned}$$

and this implies that $T^t z \in \mathcal{N}$.

Largest possible: Let $\mathcal{N}_2 \subset \ker(C)$ be another T-invariant subspace. The invariance means that $T^t z \in \mathcal{N}_2 \subset \ker(C)$ for all $t \geq 0$, so $CT^t z = 0$ for all $z \in \mathcal{N}_2$ and all $t \geq 0$. This implies that $\mathcal{N}_2 \subset \mathcal{N}$. $\square$.

Lemma 4.5.4 *(4.1.19) The reachable subspace $\overline{\mathcal{R}}$ is the smallest closed T-invariant subspace that contains* $\operatorname{range}(\mathcal{B})$.

PROOF. *Invariance:* Let $z \in \mathcal{R}$. Then $z = \mathcal{B}_\tau u = \int_{-\tau}^0 T^{-s}Bu(s)\,ds$ for some $\tau > 0$ and some u.

$$\begin{aligned} T^t z = T^t \int_{-\tau}^0 T^{-s}Bu(s)\,ds &= \int_{-\tau}^0 T^{t-s}Bu(s)\,ds \\ = \int_{-\tau-t}^{-t} T^{-r}Bu(r+t)\,ds &= \mathcal{B}^{t+\tau}v \subset \mathcal{R}, \end{aligned}$$

where$v(s) = u(s+t)$ for $s \in [-t-\tau, -t]$ and $v(s) = 0$ for $s > -t$. Thus, $\mathcal{R}$ is invariant under T, hence so is $\overline{\mathcal{R}}$.

range (B) $\subset \overline{\mathcal{R}}$*:* We use an "approximate identity" so that the input becomes the "δ-function". Let $z \in$ range(B). Then $z = Bu_0$ for some $u_0 \in U$. Define

$$u_n(s) = \begin{cases} nu_0, & -1/n \le s \le 0. \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} z_n &= \mathcal{B}_{1/n}u_n \\ &= \int_{-1/n}^0 T^{-s}Bu_n(s)\,ds \qquad (\in \mathcal{R}) \\ &= n\int_{-1/n}^0 T^{-s}Bu_0\,ds \qquad (s \to -s) \\ &= n\int_0^{1/n} T^{s}Bu_0\,ds \to Bu_0 \text{ as } n \to \infty. \end{aligned}$$

Since each $z_n \in \mathcal{R}$, and $z_n \to z$, we have $z \in \overline{\mathcal{R}}$.

Smallest: If $V \supset$ range (B) is closed and T-invariant, then

$$T^{-s}Bu(s) \subset V \text{ for all } u,\ s,$$

so (if you integrate a function whose values belong to a closed subspace, then the integral belongs to the same subspace)

$$\int_{-\tau}^0 T^{-s}Bu(s)\,ds \subset V.$$

Thus, range $\mathcal{B}_\tau \subset$ V for all $\tau > 0$, so $\mathcal{R} \subset V$. As $\mathcal{R}$ is closed, also $\overline{\mathcal{R}} \subset \overline{V}$.

4.6 Infinite Time Gramians

So far we have throughout studied controllability and observability over a finite interval of fixed length τ (sections 4.1-4.4) or varying length (section 4.5). What happens with the controllability and observability maps as $\tau \to \infty$?

Recall:

$$\mathcal{B}_\tau u = \int_{-\tau}^{0} T^{-s} B u(s)\, ds$$
$$(\mathcal{C}^\tau z)(s) = C T^s z,\ 0 \leq s \leq \tau.$$

We begin with $\mathcal{C}^\tau$. Here the extension is obvious:

$$(\mathcal{C}^\infty z)(s) = C T^s z,\ 0 \leq s < \infty.$$

There is one obvious problem, to which space do we require $\mathcal{C}^\infty z$ to belong?

First solution: Interpret $\mathcal{C}^\infty$ as a mapping from Z to $L^\infty_{\text{loc}}(\mathbb{R}^+; Y)$. This is a Frechèt space.

Second solution: Interpret $\mathcal{C}^\infty$ as a mapping from Z to an "exponentially weighted L^2-space

$$L^2_\omega(\mathbb{R}^+; Y) = y \in L^2_{\text{loc}}(\mathbb{R}^+; Y) \mid \int_0^\infty |e^{-\omega t} y(x)\, dt < \infty\}.$$

This is a Hilbert space.

Third solution: Require that the class of systems under consideration satisfies $\mathcal{C}^\infty z \in L^2(\mathbb{R}^+; Y)$ for all $z \in Z$. This is true if, for example the system is exponentially stable. (It is also true if $C = 0$ and the system is unstable.)

Here we choose the *third solution*, and we require the system to be exponentially stable. (See [Sta05] for the others.)

In the same way the the controllability map $\mathcal{B}_\tau$ is extended to

$$\mathcal{B}_\infty u = \int_{-\infty}^{0} T^{-s} B u(s)\, ds.$$

The interpretation is that the input is active from time $-\infty$ on, and we look at the state at time zero. We have the "same" three alternatives as before.

First alternative: Allow only input functions that vanish outside of a finite interval.

Second alternative: Allow input functions in $L^2_\omega(\mathbb{R}^-;Y)$ where ω is bigger than the growth rate of the system.

Third alternative: Suppose that the system is exponentially stable, and allow input functions in $L^2(\mathbb{R}^-;U)$.

Again we choose the third alternative (see [Sta05] for the others).

Warning. [CZ95] uses $\mathcal{B}^\infty u = \int_0^\infty T^s Bu(s)\, ds$ instead!

Definition 4.6.1 (4.1.20) Let $\Sigma(A,B,C,-)$ be exponentially stable. Then we define

(i) The **infinite-time controllability** (or **reachability**) map $\mathcal{B}_\infty : L^2(\mathbb{R}^-;U) \mapsto Z$ is

$$\mathcal{B}_\infty u = \int_{-\infty}^0 T^{-s} Bu(s)\, ds.$$

(ii) The **infinite-time observability map** $\mathcal{C}^\infty : Z \mapsto L^2(\mathbb{R}^+;Y)$ is

$$(\mathcal{C}^\infty z)(s) = CT^s z,\ 0 \le s < \infty$$

(iii) The **infinite-time controllability Gramian** $L_B \in \mathcal{L}(Z)$ is

$$L_B = \mathcal{B}_\infty \mathcal{B}_\infty^*$$

(iv) The **infinite-time observability Gramian** $L_C \in \mathcal{L}(Z)$ is

$$L_C = (\mathcal{C}^\infty)^* \mathcal{C}^\infty.$$

Lemma 4.6.2 *(4.1.21) The above maps are bounded linear maps between the indicated spaces.*

PROOF. Easy. See [CZ95] □

Lemma 4.6.3

$$L_C z = \int_0^\infty T^{*s} C^* C T^s z\, ds$$

and

$$L_B z = \int_0^\infty T^s B B^* T^{*s} z\, ds$$

for all $z \in Z$. These integrals converge absolutely (even exponentially) for all $z \in Z$.

PROOF. We have $(\mathcal{C}^\infty z)(s) = CT^s z$, $0 \le s < \infty$, so for all $y \in L^2(\mathbb{R}^+; Y)$,

$$\langle y, \mathcal{C}^\infty z\rangle_{L^2} = \int_0^\infty \langle y(s), CT^s z\rangle_Y \, ds = \int_0^\infty \langle T^{*s} C^* y(s), z\rangle_Z \, ds$$
$$= \left\langle \int_0^\infty T^{*s} C^* y(s) \, ds, z \right\rangle_Z = \langle (\mathcal{C}^\infty)^* y, z\rangle_Z.$$

Therefore $(\mathcal{C}^\infty)^*$ is the map

$$y \mapsto \int_0^\infty T^{*s} C^* y(s) \, ds, \; y \in L^2(\mathbb{R}^+; Y).$$

This together with the definition of $\mathcal{C}^\infty$ gives

$$L_C z = (\mathcal{C}^\infty)^* \mathcal{C}^\infty z = \int_0^\infty T^{*s} C^* C T^s z \, ds.$$

A similar computation shows that

$$L_B z = \mathcal{B}_\infty \mathcal{B}_\infty^* z = \int_{-\infty}^0 T^{-s} B B^* (T^{-s})^* z \, ds \overset{(s \to -s)}{=} \int_0^\infty T^s B B^* T^{*s} z \, ds.$$

$\square$

Theorem 4.6.4 *(4.1.23) Let $\Sigma(A, B, C, -)$ be exponentially stable. Then the gramians L_C and L_B are the unique (self-adjoint) solutions of the equations*

$$\begin{cases} AL_B + L_B A^* = -BB^* & \textit{(controllability Lyapunov equation)} \\ A^* L_C + L_C A = -C^* C & \textit{(observability Lyapunov equation)} \end{cases} \tag{4.6}$$

More precisely, these equations should be interpreted in the following way: $L_B \in \mathcal{L}(Z)$ maps $Z \mapsto Z$ and $\mathscr{D}(A^*) \mapsto \mathscr{D}(A)$, $L_C \in \mathcal{L}(Z)$ maps $Z \mapsto Z$ and $\mathscr{D}(A) \mapsto \mathscr{D}(A^*)$, and

$$\begin{cases} AL_B z + L_B A^* z = -BB^* z, & z \in \mathscr{D}(A^*), \\ A^* L_C x + L_C A x = -C^* C x, & x \in \mathscr{D}(A). \end{cases}$$

Another more symmetric way to write this is to take the inner product of the first equation with $x \in \mathscr{D}(A^*)$ and the inner product of the second equation with $z \in \mathscr{D}(A)$ and write

$$\begin{cases} \langle A^* x, L_B z\rangle + \langle L_B x, A^* z\rangle = -\langle B^* x, B^* z\rangle, & x, z \in \mathscr{D}(A^*) \\ \langle Az, L_C x\rangle + \langle L_C z, Ax\rangle = -\langle Cz, Cx\rangle, & x, z \in \mathscr{D}(A). \end{cases} \tag{4.7}$$

Here we can actually even without loss of generality take $x = z$, because if these identities are true when $x = z$, then they are also true when $x \neq z$ (use the "polarization identity").

The two Lyapunov equations can be reduced to each other by means of the following lemma.

Lemma 4.6.5 *Let A be the generator of an exponentially stable semigroup. Then the infinite-time observability map $\mathcal{C}^\infty$ of $\Sigma(A, -, C)$ is related to the controllability map $\mathcal{B}_\infty$ of $\Sigma(A^*, C^*, -)$ as follows:*

$$(\mathcal{C}^\infty)^* = \mathcal{B}_\infty \Upsilon$$

where Υ is the reflection operator

$$(\Upsilon u)(s) = u(-s),\ s \in \mathbb{R}.$$

Proof. Easy.

In particular, the observability gramian $L_C = (\mathcal{C}^\infty)^* \mathcal{C}^\infty$ of $\Sigma(A, -, C)$ is equal to the controllability gramian $L_B = \mathcal{B}_\infty \mathcal{B}_\infty^* = \mathcal{B}_\infty \mathcal{R}\mathcal{R}\mathcal{B}_\infty^*$ of $\Sigma(A^*, C^*, -)$.

Proof of Theorem 4.6.4. Because of Lemma 4.6.5, it suffices to prove e.g. the claim about the observability gramian L_C. We begin by showing that (4.6) and (4.7) are equivalent. If (4.6) holds, then we get (4.7) simply by taking the inner product as explained on p. 55. Suppose that (4.7) holds. Let $x \in \mathscr{D}(A)$. Then

$$z \mapsto -\langle Cz, Cx\rangle - \langle L_C z, Ax\rangle,\ (z \in \mathscr{D}(A))$$

has an obvious extension to a bounded linear operator $z \mapsto \mathbb{C}$ (since C and L_C are bounded). Therefore also $z \mapsto \langle Az, L_C x\rangle$ has the same extension, so $L_C x \in \mathscr{D}(A^*)$, and $\langle Az, L_C x\rangle = \langle z, A^* L_C x\rangle$. Using this in (4.7) we get

$$\langle z, A^* L_C x + L_C A x + C^* C x\rangle = 0,\ z \in \mathscr{D}(A).$$

Since $\mathscr{D}(A)$ is dense in Z, this implies that

$$A^* L_C x + L_C A x + C^* C x = 0,\ x \in \mathscr{D}(A),$$

i.e., (4.6) hold. (The proof of first half of (4.6) is analogous.)

In the sequel we replace (4.6) by (4.7).

L_C satisfies (4.7): For $x, z \in \mathscr{D}(A)$ we have (use the rule for the derivative of the product)

$$\frac{d}{dt}\langle CT^t z, CT^t x\rangle_Y = \langle CT^t Az, CT^t x\rangle_Y + \langle CT^t z, CT^t Ax\rangle_Y,$$

hence

$$\int_0^\infty \underbrace{\langle CT^t Az, CT^t x\rangle_Y}_{\to 0 \text{ exponentially as } t\to\infty} \, dt + \int_0^\infty \langle CT^t z, CT^t Ax\rangle_Y \, dt$$
$$= \left[\langle CT^t z, CT^t x\rangle\right]_0^\infty = -\langle Cz, Cx\rangle.$$

The first two integrals are equal to

$$\begin{aligned} &= \int_0^\infty \langle Az, T^{*t}C^*CT^t x\rangle \, dt + \int_0^\infty \langle z, T^{*t}C^*CT^* Ax\rangle \, dt \\ &= \langle Az, L_C x\rangle + \langle z, L_C Ax\rangle \\ &= \langle Az, L_C x\rangle + \langle L_C z, Ax\rangle. \end{aligned}$$

Thus (4.7) holds.

Uniqueness: Let L be another self.adjoint solution. Put $\Delta = L_C - L$. Then

$$\langle Az, \Delta x\rangle + \langle \Delta z, Ax\rangle = 0.$$

Here we take $x = T^t x_0$ and $z = T^t z_0$, and get

$$\begin{aligned} \frac{d}{dt}\langle T^t z_0, \Delta T^t x_0\rangle &= \langle AT^t z_0, \Delta T^t x_0\rangle + \langle T^t z_0, \Delta AT^t x_0\rangle \\ &= \langle Az, \Delta x\rangle + \langle \Delta z, Ax\rangle \\ &= 0 \end{aligned}$$

so $\langle T^t z_0, \Delta T^t x_0\rangle = \text{constant} = \langle z_0, \Delta x_0\rangle$. But $\langle T^t z_0, \Delta T^t x_0\rangle \to 0$ as $t \to \infty$, so $\langle z_0, \Delta x_0\rangle = 0$ for all $z_0, x_0 \in \mathscr{D}(A)$. $\mathscr{D}(A)$ is dense, so $\Delta x_0 = 0$ for all $x_0 \in \mathscr{D}(A)$ which implies $\Delta = 0$. □

Chapter 5

Input-Output Maps

5.1 The Impulse Response and Transfer Function

We now return to th full system

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), & t \geq 0, \\ y(t) = Cz(t) + Du(t), & t \geq 0, \\ z(0) = z_0. \end{cases} \tag{5.1}$$

Definition 5.1.1 The **input-output map** of the system $\Sigma(A, B, C, D)$ is the mapping from the input function u in (5.1) to the output y in (5.1) when $z_0 = 0$.

Warning 5.1.2 Above we interpret z as a mild solution of (5.1), not as a classical solution, and we permit $u \in L^2_{\mathrm{loc}}(\mathbb{R}^+; U)$. The output y then belongs to $L^2_{\mathrm{loc}}(\mathbb{R}^+; Y)$.

Lemma 5.1.3 *The input-output map is given by*

$$u \mapsto \int_0^t CT^{t-s}Bu(s)\, ds + Du(t),\ t \geq 0.$$

PROOF. The mild solution is given by (if $z_0 = 0$)

$$z(t) = \int_0^t T^{t-s}Bu(s)\, ds,\ t \geq 0,$$

and $Cz(t) + Du(t) =$ the formula given above. □

Definition 5.1.4 The **impulse response** of $\Sigma(A, B, C, D)$ is the distribution

$$D\delta_0 + CT^t B,$$

where δ_0 is the δ-function at zero and $CT^t B$ is a continuous function. In particular, if $D = 0$, then the impulse response is the function $t \mapsto CT^t B$.

Interpretation: If we use $u_0\delta_0$ as an input, where u_0 is a fixed vector in U and δ_0 is the δ-function at zero, then the output of the system is the distribution

$$Du\delta_0 + CT^t Bu.$$

Lemma 5.1.5 *Let $\Sigma(A, B, C, D)$ be a system and let ω be the growth bound of the semigroup T generated by A. If $u \in L^2_{\text{loc}}(\mathbb{R}^+; U)$ and*

$$\int_0^\infty \|e^{-\omega t} u(t)\|_U \, dt < \infty,$$

then the Laplace transforms of u, z and y in (5.1) converge absolutely in the half-plane $\Re(s) \geq \omega$, and

$$\begin{aligned} \hat{z}(s) &= (s - A)^{-1}(z_0 + B\hat{u}(z)), \ \Re(s) \geq \omega, \\ \hat{y}(s) &= C(s - A)^{-1} z_0 + \hat{D}(s)\hat{u}(s), \ \Re(s) \geq 0, \end{aligned}$$

where $\hat{D}(s) = C(s - A)^{-1}B + D$.

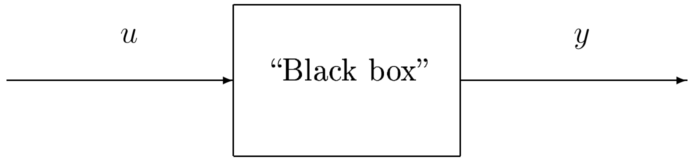
Proof. Easy, but we have no time to present it. See [CZ95] or [Sta05].

Definition 5.1.6 The operator-valued function $\hat{D}(s)$ (with values in $\mathcal{L}(U; Y)$), defined on the half-plane $\Re(s) > \omega_A$(the growth bound of the system) is called the **transfer function** of $\Sigma(A, B, C, D)$.

The course on "transfer functions" (which is a continuation of this course) will discuss these transfer functions in detail. They are very important if we look at the system from an input-output point of view.

5.2 Realizations

It will be shown in the course on "transfer functions" that under very general conditions, the relationship between the "input" and the "output" of a "**black box**" is determined by a "transfer function" G:



The black box is assumed to be

- linear
- time-invariant
- causal.

Then there is an operator-valued function G, defined on some right-half plane, with values in $\mathcal{L}(U;Y)$, such that the Laplace transform $\hat{y}$ of the output is given by

$$\hat{y}(s) = G(s)\hat{u}(s)$$

($\Re(s)$ large enough), where $\hat{u}$ is the Laplace transform of u.

Definition 5.2.1 Let G be an analytic $\mathcal{L}(U;Y)$-valued function defined on some right half-plane $\Re(s) > \omega$. We call the system $\Sigma(A,B,C,D)$ a **realization** of G if the transfer function $\hat{D}$ of $\Sigma(A,B,C,D)$ is equal to G. This realization is minimal if $\Sigma(A,B,-)$ is (approximately) controllable and $\Sigma(A,-,C)$ is (approximately) observable.

More about realizations in the course in "Transfer Functions"

The End

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