ANALYSIS II, Homework 10

Due Wednesday 4.12.2013. Please hand in written answers for credit.

1. Let H be a Hilbert space and let $F : H \to \mathbb{C}$ be a continuous linear functional. Show that the linear subspace $\{x \in H : \ F(x) = 0\}$ is closed.

2. Compute the Fourier series of the function $f(x) = x, \ x \in [-\pi, \pi]$. Moreover, use Bessel's equality to compute

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$,

(a) $\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2}$.

3. Use the iterative procedure on which the Banach fixed point theorem is based to find a power series solution of the differential equation

$$y'(x) = y(x) + x, \ y(0) = 1.$$

Hint: First integrate this equation to get

$$y(x) = y(0) + \int_0^x y'(t) \ dt = 1 + \int_o^x [y(t) + t] \ dt,$$

and then start to iterate this equation.

4. Let (X, d) be a metric space, and let $f : X \to X$ be a function satisfying

$$(\star) \quad d(f(x), f(y)) < d(x, y) \text{ if } x \neq y.$$

(a) Show that f is continuous.

(b) Can f have more than one fixed point?

(c) Give an example of a function f on some metric space (X, d) which satisfies $(\star)$ but does not have a fixed point.