ANALYSIS II, Homework 5

Due Wednesday 16.10.2013. Please hand in written answers for credit.

1. Let E be a normed space. A set $A \subset E$ is called *convex*, if for each λ with $0 \le \lambda \le 1$ we have:

$$x, y \in A \quad \Rightarrow \quad \lambda x + (1-\lambda) y \in A.$$

Show that $\{x \in E : ||x|| \le r\}$, where $r > 0$, is a convex set in E.

2. Show that in an n-dimensional vector space E, for any norm $||\cdot||$ on E we have that there is $0 < M < \infty$ such that

$$||x|| \le M \; ||x||_\infty \; \text{ for all } \; x \in E.$$

3. Let E be a normed space. True or false? If true prove it, if false explain why:

(i) $||x||^a \le ||x||^b$ for $a < b$ and for any $x \in E$.

(ii) $||x||^a \le c(1 + ||x||^b)$ for some constant $c > 0$ independent of x, for any $b \ge a > 0$ and any $x \in E$.

(iii) $||x||^a ||y||^b \le ||x||^{a+b} + ||y||^{a+b}$ for any $a, b \ge 0$ and any $x, y \in E$.

4. Let $(x_n)_n$ be a sequence in l^2 defined by $x_n^k = \frac{1}{n+k}$ (notation: x_n^k is the k-th element of the n-th sequence). Show that $(x_n)_n$ converges to $0 \in l^2$.

5. Let X be a vector space and $p : X \to [0, \infty)$ a function satisfying:

- $p(x) = 0$ if and only if $x = 0$, and
- $p(\lambda x) = |\lambda| p(x)$ for all $x \in X$ and $\lambda \in \mathbb{R}$.

Show that p is a norm if and only if the set $\{x \in E : p(x) \le 1\}$ is convex.

6. Let (X, d) be a metric space. Let A and B be two non-empty, closed subsets of X such that $A \cap B = \emptyset$. Show that there is a continuous function $f : X \to [0, 1]$ with $f(A) = \{0\}$ and $f(B) = \{1\}$.