35th Finnish Summer school in probability and statistics

Bibliography on the lecture
An introduction to rough paths

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In this lecture, we give a short introduction to the theory of rough paths initiated by Terry Lyons at the end of the 90's. This theory provides us with a way to define integrals and differential equations driven by irregular paths. It is particularly suitable to define stochastic integrals and stochastic differential equations driven by other processes than the Brownian motion or semi-martingales. This theory has for example proved to be fruitful when used with Gaussian processes, for numerical computations (the so-called cubature formula) and now, and more recently for dealing with some stochastic partial differential equations. In this lecture, we focus on the main concepts and results of this theory (Young integrals, the sewing lemma, defining integrals and differential equations driven by rough paths, ...).

1 General references

⋆ The original article [20]
⋆ Books and surveys from T. Lyons [19, 21]
⋆ Other lectures and surveys [1, 2, 4, 7, 9, 13, 15, 17, 18]
⋆ The Davie's approach (Euler scheme) [8]
⋆ The Friz-Victoir's approach (Combining geodesics and the Davie's approach) [12, 14]
⋆ The Gubinelli's approach (controlled rough paths) [16]
⋆ The Feyel-de la Pradelle's approach (Short proof of the sewing lemma) [7, 10, 11]
⋆ The Bailleul's approach (flows) [3]

Other references on related subjets:
⋆ On stochastic flows [5]
⋆ On the Magnus formula for linear differential equations [6]

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2 General results about rough differential equations (RDE) and rough paths

These results do not necessarily appear in the reference given in § 1.

⋆ Young differential equations [8, 28, 33]
⋆ Lipschitz continuity of the Itô map [29, 30]
⋆ Differentiability of Itô map [23, 12, 32, 33, 36]
⋆ Global existence [12, 29, 30]
⋆ Geometric and non-geometric rough paths [24, 26, 31]
⋆ Generic construction of rough paths above a given path [34, 37]
⋆ Convergence of Euler scheme [8, 14]
⋆ Linear RDE [22]
⋆ Branching rough paths [25, 26, 37]
⋆ Signature of a path [27, 35]

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3 Fields of applications

The literature on rough paths theory is now important. As there are too many references, I just give the name of some authors, probably missing many ones...

⋆ Application to stochastic processes
- Fractional Brownian motion L. Coutin, Y. Hu, D. Nualart, S. Tindel, A. Deya, J. Unterberger, ...
- Brownian motion and semi-martingales: L. Coutin, P. Friz, A. Lejay, E.-M. Sipilainen, N. Victoir, ...

  - Gaussian processes: P. Friz, M. Hairer, S. Tindel, N. Victoir, ...
  - Processes generated by divergence-form operators : P. Friz, A. Lejay, P. Friz
  - Free Brownian motion: N. Victoir
  - ...
- ⋆ Cubature: C. Bayer, D. Crisan, T. Lyons, J. Teichmann, N. Victoir, ...
- ⋆ Monte Carlo Numerical methods: C. Bayer, P. Friz, A. Lejay, G. Pagès S. Riedel, V. Reutenauer, J. Schoenmakers, A. Sellami, ...
- ⋆ Stochastic Partial Differential Equations: A. Deya, M. Gubinelli, M. Hairer, S. Tindel, ...
- ⋆ Backward Stochastic Differential Equations: J. Diehl, P. Friz.
- ⋆ Malliavin calculus: F. Baudoin, T. Cass, M. Hairer, S. Tindel, ...
- ⋆ Laplace expansion, large deviation: Y. Inahama, H. Kawabi, M. Ledoux, T. Lyons, A. Millet, Z. Qian, M. Sanz-Solé. ...
- ⋆ Non-linear filtering: D. Crisan, P. Friz, T. Lyons, ...
- ⋆ Stochastic control: J. Diehl
- ⋆ Quantum Field Theory: J. Unterberger
- ⋆ Non-linear Fourier analysis : K. Hara, T. Lyons
- ⋆ Homogenization: A. Lejay, T. Lyons
- ⋆ ...

4 Historical references

- ⋆ Pathwise integration and numerical methods [38, 44–47]
- ⋆ Young integrals [46, 48]
- ⋆ Chen series and applications [39–43]

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