On optimal stopping of risk process

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Abstract. The following problem in risk theory is considered. An insurance company, endowed with an initial capital $a > 0$, receives insurance premiums and pays out successive claims from two kind of risks. The losses occur according to marked point process. At any moment the company may broaden or narrow down the offer, what entail the change of the parameters of the underlying risk process. These changes concern the rate of income, the intensity of renewal process and the distribution of claims. Our goal is to find the best moment for changes which is the moment of maximal value of the capital assets. Based on the representation of the stopping times for the piecewise deterministic processes and a dynamic programming method the solution is derived for the finite horizon model. It opens a method of optimal management the insurance risks.
Keywords Risk reserve process; optimal stopping; dynamic programming; multivariate point process; capital assets

1 Introduction

Let us consider the insurance company having an initial capital $a > 0$ which insures two kind of risks. The i-th risk makes the stream of insurance premiums with constant rate c_i and pays out successive claims, which are representing by $i.i.d.$ random variables $X_{i,1}, X_{i,2}, \ldots$ with cumulative distribution function H_i. The losses related to the i-th risk occur according to the renewal process $\{N_i(t), t \geq 0\}$, where $N_i(t)$, $i = 1, 2$, are the number of claims up to the time t in the stream of the risk i. The renewal processes are mutually independent and they are independent of the sequence of claims. The 2-vector point process $(N_1(t), N_2(t))$, $t \geq 0$ can be represented also by a sequence of random variables T_n taking values in $[0, \infty]$ such that $T_0 = 0$, $T_n < \infty$ then $T_n < T_{n+1}$ for $n \in \mathbb{N}$, and a sequence of $\{1, 2\}$-valued random variables Z_n for $n \in \mathbb{N} \cup \{0\}$ (see Brémaud [3] Ch. II, Jacobsen [8]). Denote $N(t) = N_1(t) + N_2(t)$. Let us assume that there are $i.i.d.$ r.v. $X_{i,n}$ with continuous, cumulative distribution function F_i. Define the balance between premium and collection of payoffs covering the claims $R_i(t) = c_i t - \sum_{s=1}^{N_i(t)} X_{i,s}$ in the risk i. The both risks are parts of the capital assets U_t, $t \geq 0$, of the insurance company $U_t = a + \sum_{i=1}^{2} R_i(t)$. Let $g(u, t) = g_1(u)\mathbb{I}_{\{t \geq 0\}}$, where g_1 is a utility function. The return at time t is

$\{Z_t\}_{\{t\geq 0\}}$ and it is given by

$$Z_t = g(U_t, t_0 - t)\prod_{j=0}^{N(t)} \mathbb{I}_{\{U_{T_j}>0\}} = g(U_t, t_0 - t)\mathbb{I}_{\{U_s>0, s\leq t\}} \tag{1}$$

The optimal stopping problem for the process $\{Z_t\}_{\{t\geq 0\}}$ is investigated.

Assumption 1. *The utility function g_1 is bounded, continuous, nondecreasing and differentiable.*

Let $\mathcal{T}$ be a set of stopping times with respect to σ-fields $\{\mathcal{F}_t\}$, $t \geq 0$, defined by the multivariate point process and the sequence of claims. The restricted sets of stopping times $\mathcal{T}_{n,K} = \{\tau \in \mathcal{T} : \tau \geq 0, T_n \leq \tau \leq T_K\}$ for $n \in \mathbb{N}$, $n < K$ are subsets of $\mathcal{T}$. In the optimization problems formulated for the risk process the representation of the stopping times according to the following way plays a crucial role (see [3] Appendix A.2, cf. [4] Th. A2.3). Let us denote $\mathcal{T}_{i,n,K} = \{\tau \in \mathcal{T}_K : T_{i,n} \leq \tau \leq T_K\}$. If $\tau \in \mathcal{T}_{i,n,K}$, then there exists a positive, $\mathcal{F}_{i,n}$-measurable, random variable $R_{i,n}$ such that

$$\tau \wedge T_{j,N_j(T_{i,n})+1} \wedge T_{i,n+1} = (T_{i,n} + R_{i,n}) \wedge T_{j,N_j(T_{i,n})+1} \wedge T_{i,n+1}, \text{ a.s..}$$

The aim of the decision maker is to find the stopping time $\tau^\star_K \in \mathcal{T}_K$ such that

$$\mathbf{E}Z_{\tau^\star_K} = \sup_{\tau\in\mathcal{T}_K} \mathbf{E}Z_\tau. \tag{2}$$

In order to find the optimal stopping time $\tau^\star_K$ according to the definition (2), we first consider optimal stopping times $\tau^\star_{i,n,K}$, such that

$$\Gamma_{i,n,K} = \mathbf{E}(Z_{\tau^\star_{i,n,K}}|\mathcal{F}_{i,n}) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{i,n,K}} \mathbf{E}(Z_\tau|\mathcal{F}_{i,n}), \tag{3}$$

for $i \in \{1, 2\}$, $n = 0, 1, \ldots, K$ and using backward induction by i and n as in dynamic programming, we will obtain $\tau^\star_K = \tau^\star_{0,K} = \tau^\star_{0,0,K}$ (the claims at moment 0 for both type of risks are equal 0 by assumption).

2 Case with fixed number of claims

Let $\mu_0 = 1$ and $\mu_n = \prod_{j=1}^n \mathbb{I}_{\{U_{T_j}>0\}}$. Then $\Gamma_{i,K,K} = Z_{T_{i,K}} = g(U_{T_{i,K}}, t_0 - T_{i,K})\mu_K$. Note that the sum of claims can be expressed as

$$\sum_{n=0}^{N_1(t)} X_{1,n} + \sum_{n=0}^{N_2(t)} X_{2,n} = a + (c_1 + c_2)t - U_t. \tag{4}$$

Let us define for $\xi > 0$ such that there is no jump between t and $t + \xi$

$$d(t, \xi, U_t) = U_{t+\xi} - U_t = (c_1 + c_2)\xi \tag{5}$$

then we have

$$\mu_{i,K} = \mu_{i,K-1}\mathbb{I}_{\{U_{T_{i,K-1}}+d(T_{i,K-1},\zeta_{i,K},U_{T_{i,K-1}})-X_{i,K}>0\}}. \tag{6}$$

Similarly as in [11], Theorem 1, from (4) and from (5) we get the dynamic programming equations for $\{\Gamma_{i,n,K}\}_{n=0}^K$, $i = 1, 2$.

Theorem 1. *For the given horizon t_0, $\{1,2\}$-marked renewal processes and fixed number of claims K, the optimal expected value of the stopping problem with the restricted sets of stopping times $\mathcal{T}_{i,n,K}$ fulfills the recursive equations:*

(i) *For $n = K-1, K-2, \ldots, 0$, $i = 1, 2$,*

$$\begin{aligned}\Gamma_{i,n,K} &= \operatorname{ess\,sup}\big\{\mu_n \bar{F}_i(\xi)\bar{F}_j^{T_{i,n}}(\xi + T_{i,n} - T_{j,N_j(T_{i,n})}) \\ &\quad \times g\left(U_{T_{i,n}} + d(T_{i,n}, \xi, U_{T_{i,n}}), t_0 - T_{i,n} - \xi\right) \\ &\quad + \mathbf{E}(\mathbb{I}_{\{\xi \ge \zeta_{j,N_j(T_{i,n})+1} + T_{j,N_j(T_{i,n})} - T_{i,n}, \bar{A}_{i,n}\}}\Gamma_{j,N_j(T_{i,n})+1,K}|\mathcal{F}_{i,n}) \\ &\quad + \mathbf{E}(\mathbb{I}_{\{\xi \ge \zeta_{i,n+1}, A_{i,n}\}}\Gamma_{i,n+1,K}|\mathcal{F}_{i,n}) : \xi \ge 0 \text{ is } \mathcal{F}_{i,n}\text{-measurable}\big\} \quad a.s.,\end{aligned}$$

where $\bar{F}_i = 1 - F_i$ are the survival functions and $A_{i,n} = \{\omega : \zeta_{i,n+1} = \zeta_{i,n+1} \wedge (\zeta_{j,N_j(T_{i,n})+1} + T_{j,N_j(T_{i,n})} - T_{i,n})\}$.

(ii) *For $n = K, K-1, \ldots, 0$, $i, j \in \{1,2\}$, $i \ne j$, $\Gamma_{i,n,K} = \mu_n \gamma_{i,K-n}(U_{T_{i,n}}, T_{j,N_j(T_{i,n})}, T_{i,n})$ a.s., where the sequence of functions $\{\gamma_{i,n}(u,s,t), u \in \Re, s,t \ge 0, s \le t\}$, using (5), (4) and (6) is defined as follows:*

$$\begin{aligned}\gamma_{i,0}(u,s,t) &= g(u, t_0 - t), \\ \gamma_{i,k}(u,s,t) &= \sup_{r \ge 0}\Big[\bar{F}_i(r)\bar{F}_j^t(t + r - s)g(u + cr, t_0 - t - r) \\ &+ \int_0^r dF_i(\xi)\int_{\xi+t-s}^{\infty} dF_j(\eta)\int_0^{u+c\xi}\gamma_{i,k-1}(u + c\xi - x, t + \xi)\, dH_i(x) \\ &+ \int_0^{r+t-s} dF_j(\eta)\int_{\eta+s-t}^{\infty} dF_i(\xi)\int_0^{u+c(\eta+s-t)}\gamma_{j,k-1}(u + c(\eta + s - t) - x, s + \eta)\, dH_j(x)\Big]\end{aligned}$$

$j = 1, 2, \ldots$, $i = 1, 2$ and $c = c_1 + c_2$.It can be rewritten in simplified form

$$\begin{aligned}\gamma_{i,0}(u,s,t) &= g(u, t_0 - t), \\ \gamma_{i,k}(u,s,t) &= \sup_{r \ge 0}\Big[\bar{F}_i(r)\bar{F}_j^t(t + r - s)g(u + cr, t_0 - t - r) \\ &+ \int_0^r dF_i(\xi)\bar{F}_j^t(\xi + t - s)\int_0^{u+c\xi}\gamma_{i,k-1}(u + c\xi - x, t + \xi)\, dH_i(x) \\ &+ \int_0^{r+t-s} dF_j(\eta)\bar{F}_i^s(\eta + s - t)\int_0^{u+c(\eta+s-t)}\gamma_{j,k-1}(u + c(\eta + s - t) - x, s + \eta)\, dH_j(x)\Big]\end{aligned}$$

Remark 1. *The above equations differ from the ones in Theorem 1 in [11] as a result of a different form of the capital assets process U_t. The optimal value of the problem with the given number of claims K is equal $\Gamma_{1,0,K} = \Gamma_{2,0,K}$.*

The next step is to find the optimal stopping time τ_K^*. To do this we should analyze the properties of the sequence of functions $\{\gamma_{i,n}, n \ge 0, i \in \{1,2\}\}$. Let $B = B[(-\infty, +\infty) \times [0, +\infty) \times [0, +\infty)]$ be the space of all bounded and continuous functions with the norm $||\delta|| = \sup_{u,s,t}|\delta(u,s,t)|$ and let $B^0 = \{\delta : \delta(u,s,t) = \delta_1(u,s,t)\mathbb{I}_{\{s \le t \le t_0\}}$ and $\delta_1 \in B\}$. One should notice that the functions $\{\gamma_{i,n}, n \ge 0\}$ are included in B^0. For each $\vec{\delta} = (\delta_1, \delta_2) \in B^0 \times B^0$ and any $u \in \Re$,

$s, t, r \geq 0$ let

$$\begin{aligned}(\vec{\phi}_{\vec{\delta}}(r,u,s,t))_i &= \bar{F}_i(r)\bar{F}_j^t(t+r-s)g(u+cr,t_0-t-r) \\ &+ \int_0^r dF_i(\xi)\bar{F}_j^t(\xi+t-s)\int_0^{u+c\xi}\delta_i\left(u+c\xi-x,t+\xi\right)dH_i(x) \\ &+ \int_0^{r+t-s} dF_j(\eta)\bar{F}_i^s(\eta+s-t)\int_0^{u+c(\eta+s-t)}\delta_j\left(u+c(\eta+s-t)-x,s+\eta\right)dH_j(x).\end{aligned}$$

From the properties of the cumulative distribution function F. we know that $\vec{\phi}_{\vec{\delta}}(r,u,s,t)$ has at most a countable number of points of discontinuity according to r and is continuous according to (u,s,t) in the case of $g_1(\cdot)$ being continuous and $t \neq t_0 - r$. Therefore, for further considerations we assume that the function $g_1(\cdot)$ is bounded and continuous.

For each $\vec{\delta} \in B^0 \times B^0$ let

$$(\vec{\Phi}\vec{\delta})_i(u,s,t) = \sup_{r\geq 0}\{(\vec{\phi}_{\vec{\delta}})_i(r,u,s,t)\} \text{ for } i \in \{1,2\}. \tag{7}$$

Lemma 2. *For each $\vec{\delta} \in B^0 \times B^0$ we have*

$$(\vec{\Phi}\vec{\delta})_i(u,s,t) = \max_{0\leq r\leq t_0-t}\{(\vec{\phi}_{\vec{\delta}})_i(r,u,s,t)\} \in B^0$$

and there exists a function $r_{\delta_i}(u,s,t)$ such that $(\vec{\Phi}\vec{\delta})_i(u,s,t) = (\vec{\phi}_{\vec{\delta}})_i(r_{\delta_i}(u,s,t),u,s,t)$.

In subsequent considerations more properties of $\vec{\Phi}$ will be presented.

For $k = 1, 2, \ldots$, $i \in \{1,2\}$ and $u \in \Re$, $s, t \geq 0$, $\gamma_{i,k}(u,s,t)$ may be expressed as follows

$$\gamma_{i,k}(u,s,t) = (\vec{\gamma}_k)_i(u,s,t) = \begin{cases} (\vec{\Phi}\vec{\gamma}_{k-1})_i(u,s,t) & \text{if } u \geq 0 \text{ and } s \leq t \leq t_0, \\ 0 & \text{otherwise,}\end{cases}$$

and from Lemma 2 there exist functions $r_{(\vec{\gamma}_{k-1})_i}(u,s,t)$ such that

$$\gamma_{i,k}(u,s,t) = \begin{cases} (\vec{\phi}_{\vec{\gamma}_{k-1}})_i(r_{(\vec{\gamma}_{k-1})_i}(u,s,t),u,s,t) & \text{if } u \geq 0 \text{ and } s \leq t \leq t_0, \\ 0 & \text{otherwise.}\end{cases}$$

To specify the form of the optimal stopping times $\tau^*_{i,n,K}$, we need to define the following random variables $R^*_{i,s} = r_{\vec{\gamma}_{K-s+1}}(U_{T_{i,s}}, T_{j,N_j(T_{i,s})}, T_{i,s})$ and $\sigma_{i,n,K} = K \wedge \inf\{s \geq n :\ R^\star_{i,s} < S_{i,s+1}\}$.

Finally in Corollary 3 we present the form of the optimal stopping time.

Corollary 3. *Let*

$$\tau^*_{i,n,K} = T_{i,\sigma_{i,n,K}} + R^\star_{i,\sigma_{i,n,K}} \quad \textit{and} \quad \tau^\star_K = \tau^*_{0,K} = \tau^*_{i,0,K},$$

then for all $0 \leq n \leq K$ the following hold

$$\Gamma_{i,n,K} = \mathbf{E}(Z_{\tau^*_{i,n,K}}|\mathcal{F}_{i,n}) \ \textit{a.s.} \quad \textit{and} \quad \Gamma_{0,K} = \mathbf{E}(Z_{\tau^*_{i,0,K}}) = (\vec{\gamma}_K)_i(a,0,0),$$

which means $\tau^*_{i,n,K}$ and $\tau^\star_K$ are optimal stopping times in the classes $\mathcal{T}_{i,n,K}$ and $\mathcal{T}_{0,K}$ respectively.

3 Infinite number of claims

In this Section we consider the case of infinite number of claims and we find stopping time τ^*, which is optimal in the class $\mathcal{T}$. Let us restrict on cumulative distribution functions such that

Assumption 2. *$F_i(t_0) < 1$ for $i = 1, 2$.*

The following lemma (see [6]) will play the important role in our considerations.

Lemma 4. *The operator $\vec{\Phi} : B^0 \times B^0 \to B^0 \times B^0$ defined by (7) is a contraction.*

As $\vec{\gamma}_0 \in B^0 \times B^0$ we conclude that $\vec{\gamma}_k \in B^0 \times B^0$ for all k, therefore we can use the Banach fixed point theorem (see e.g. [7]) and obtain the following lemma

Lemma 5. *There exists $\vec{\gamma} \in B^0 \times B^0$ such that*

$$\vec{\gamma} = \vec{\Phi}\vec{\gamma} \text{ and } \lim_{K\to\infty} \|\vec{\gamma}_K - \vec{\gamma}\| = 0.$$

Corollary 6. *$\vec{\gamma}$ is uniform limit of $\vec{\gamma}_K$, when K tends to infinity.*

The consideration of Sections 2 and 3 leads to the following formulation of the optimal strategy after the change of parameters in the risk process.

Theorem 2. *If the function g fulfils Assumption 1, F_i fulfils Assumption 2 and has the density function f_i, $i = 1, 2$, then*

(i) *for $n \in \mathbb{N}$ the limit $\tau^*_{i,n} = \lim_{K\to\infty} \tau^*_{i,n,K}$ a.s., for $i = 1, 2$, exists and $\tau^*_{i,n}$ is an optimal stopping rule in the set $\mathcal{T} \cap \{\tau \geq T_{i,n}\}$,*

(ii) *$E(Z_{\tau^*_{i,n}}|\mathcal{F}_{i,n}) = \mu_{i,n}\gamma_{i,n}(U(T_{i,n}), T_{j,N_j(T_{i,n})}, T_{i,n})$ a.s..*

At the end of this Section we notice that optimal stopping time is equal to $\tau^* = \tau^*_{i,0} \in \mathcal{T}$, where $\tau^*_{i,0} = \lim_{K\to\infty} \tau^*_{i,0,K}$. The value function of the optimal stopping problem is given by $E\{Z_{\tau^*}\} = \mu_{i,0}\gamma(U(0, t_0), 0, 0)$.

4 Final remarks

The considered processes allow to describe the reserves of the insurance companies. The management decisions concerning the portfolio of risks change parameters of the processes. The main idea is to find the good moment to manage the assets. Various approaches have been taken into account which are subject of the research (see Azcue and Roberts [1], Ferenstein and Sierociński [6], Karpowicz and Szajowski [9], Muciek [11], Muciek and Szajowski [12], Ferenstein and Pasternak-Winiarski [5], Rolski et al. [13], Schöttl [14]). The results are an extension of optimal stopping problems formulated as by Boshuizen [2] for the multivariate renewal processes and for the risk process by Karpowicz and Szajowski [10].

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