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Peacocks and Bougerol's identity

1)

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The following notes constitute a summary of the lecture I delivered at SF-180 - in honor of Paavo Salminen, Esko Valkeila, and Esa Nummelin.

I have known Paavo personally since 1986, when I gave a course at the Lahti Spring School. Since then, we have visited each other several times, and wrote a number of joint papers.

1. Peacocks as "traces" of martingales.

It is a consequence of Jensen's inequality that if $(M_t)_{t\geq 0}$ is a martingale, and $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is a convex function, then:

(1) for $s < t$, $\quad E[\psi(M_t)] \geq E[\psi(M_s)]$

Thus, one says that $(M_t, t\geq 0)$ is a process increasing in the convex order; the acronym is PCOC (in French).

As a pun, we termed these processes peacocks and a volume by F. Hirsch, C. Profeta, B. Roynette and I has now appeared: Peacocks and associated martingales

Bocconi - Springer (2011)

What is remarkable is that if $(X_t)_{t\geq 0}$ is a peacock, then there exists at least one martingale $(M_t)_{t\geq 0}$ such that:

$$(2) \qquad \boxed{X_t \overset{(1.d)}{=} M_t},$$

where (1.d) means equality in law for 1-dimensional distributions.

This is a theorem due to Kellerer (1972), who built on several previous partial results.

The volume on Peacocks already mentioned systematically searches for a martingale (M_t) satisfying (2), given a peacock (X_t).

The first peacock I encountered is the arithmetic average of geometric Brownian motion:

$$(3) \qquad \boxed{\mathcal{A}_t = \frac{1}{t}\int_0^t ds \exp\left(B_s - \frac{s}{2}\right)}$$

In fact, it was shown by Carr - Ewald - Xiao (December 2008) that $(\mathcal{A}_t, t\geq 0)$ is a peacock.

I searched for several months a martingale $(M_t)_{t\geq 0}$ such that, (4) $\mathcal{A}_t \overset{(1.d)}{=} M_t$,

and finally, D. Baker and I (2009) published the following result:

$$\boxed{M_t = \int_0^1 du \exp\left(W_{u,t} - \frac{ut}{2}\right), \ t\geq 0,}$$

where $(W_{u,t})_{u,t \geq 0}$ is a standard Brownian ~~sh~~sheet

is a martingale with respect to: $\mathcal{W}_t \equiv \sigma\{W_{u,s};\ u \geq 0, s \leq t\}$

and it satisfies (4). This has been the very beginning of the book on Peacocks and we call $(\mathcal{A}_t, t \geq 0)$ the guiding example (of peacocks).

2. Another interesting peacock : $(\sinh(B_t), t \geq 0)$

It has been shown in 1983 by Bougerol (: Ann. IHP) that:

$$(5) \qquad \boxed{\sinh(B_t) \overset{(1\text{-}d)}{=} \beta_{A_t}} ,$$

where $A_t = \int_0^t ds \exp(2B_s)$, and $(\beta_u, u \geq 0)$ is a Brownian motion independent of $(B_s, s \geq 0)$ our original Brownian motion.

Thus, (5) shows that $(\sinh(B_t), t \geq 0)$ is a peacock.

Let us recall ~~the~~ the proof of (5), as given by Alili–Dufresne–Yor (1997).

- If we denote $S_t = \sinh(B_t)$, then Itô's formula shows that:

$$(6) \qquad \boxed{S_t = \int_0^t \sqrt{1 + S_s^2}\, dB_s + \frac{1}{2} \int_0^t S_s\, ds.}$$

— On the other hand, by Dubins–Schwarz theorem, 4)

the process : $\left(\int_0^t \exp(B_s)\, d\beta_s ,\ t \geq 0 \right)$ is distributed

as $\left(\beta_{A_t} , t \geq 0 \right)$. Moreover by time-reversal, one has:

$$(7) \qquad \int_0^t \exp(B_s)\, d\beta_s \overset{(l.d)}{=} \exp(B_t) \int_0^t \exp(-B_s)\, d\beta_s$$

Now, let us consider the RHS of (7), and call it $(\Sigma_t, t \geq 0)$

then, again applying Itô's formula, we obtain:

$$(8) \qquad \boxed{\Sigma_t = \beta_t + \int_0^t \Sigma_s\, dB_s + \frac{1}{2} \int_0^t ds\, \Sigma_s}$$

Gathering the two martingales featured in (8) we discover that there exists another brownian motion $(\delta_t, t \geq 0)$ such that:

$$(9) \qquad \Sigma_t = \int_0^t \sqrt{1 + \Sigma_s^2}\, d\delta_s + \frac{1}{2} \int_0^t ds\, \Sigma_s$$

Comparing (9) and (6), we discover that the processes

$(\Sigma_t, t \geq 0)$ and $(S_t, t \geq 0)$ are identically distributed, hence Bougerol's identity is proven and well understood.

3. A generalisation of Bougerol's identity

Several extensions of Bougerol's identity have already been obtained; see, e.g., in the Revista volume (1997) and also, more recently by Bertoin–Dufresne–Yor (2011)

Here, we propose yet another 2-parameter extension of Bougerol's identity, which indeed was started from the following simple consequence from (5):

$$(10) \qquad \sinh(|B_t|) \overset{(law)}{=} |\beta_{A_t}|$$

Hence, it may be reasonable to look at processes of the form:

$$\left(R^{(\delta)}_{A^{(\nu)}_t} \, , t \geq 0 \right)$$

where $(R^{(\delta)}_u, u \geq 0)$ is a Bessel process with dimension δ independent from

$$A^{(\nu)}_t = \int_0^t ds \exp(2 B^{(\nu)}_s)$$

where: $B^{(\nu)}_s \equiv B_s + \nu s$, Brownian motion with drift ν.

If $R^{(\delta)}_0 = 0$, we note that:

$$(11) \qquad R^{(\delta)}_{A^{(\nu)}_t} \overset{(law)}{=} e^{-B^{(\nu)}_t} R^{(\delta)}_{A^{(\nu)}_t}$$

Now, it turns out that the RHS of (11) is a diffusion, which may be related to the Jacobi processes – (see, Warren–Yor)

Thus, the extension of this diffusion:

$$X^{(\nu,\delta)}_t \overset{def}{=} \exp(-B^{(\nu)}_t) R^{(\delta)}_{A^{(\nu)}_t}$$ constitutes an extension of Bougerol's identity

References.

F. Hirsch, C. Profeta, B. Roynette, M. Yor

Peacocks and associated martingales, with explicit constructions. Bocconi University Press, Springer (2011).

D. Baker, M. Yor. A brownian sheet martingale with the same marginals as the arithmetic average of geometric brownian motion. Elec. J. Prob. 14 (52), 1532–1540 (2009).

P. Carr, C.O. Ewald, Y. Xiao: On the qualitative effect of volatility and duration on prices of Asian options. Finance Research Letters 5(3), 162–171 (September 2008).

P. Bougerol: Exemples de théorèmes locaux sur les groupes résolubles. Ann. IHP, 19 (4), p. 369–391 (1983).

L. Alili, D. Dufresne, M. Yor: Sur l'identité de Bougerol pour les fonctionnelles exponentielles du mouvement brownien avec drift. In: Exponential Functionals and principal values related to Brownian motion, Bibl. Rev. Math. Iberoamericana, Madrid (1997).

H. G. Kellerer: Markov-Komposition und eine Anwendung auf Martingale. Math. Ann 198 p. 99–122 (1972).