Independence properties of Matsumoto-Yor type and characterization of Kummer, Gamma and Beta distributions.

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1. Introduction

1.1 Relations between the GIG distributions, gamma distributions and the function $f_0(x) = 1/x$ ($x > 0$)

The GIG distribution with parameters $\mu \in \mathbb{R}$, $a, b > 0$ is the probability measure :

$$GIG(\mu, a, b)(dx) = \left(\frac{b}{a}\right)^{\mu} \frac{x^{\mu-1}}{2K_\mu(ab)} \exp\left\{ -\frac{1}{2}\left(b^2 x + \frac{a^2}{x}\right)\right\} \mathbf{1}_{(0,\infty)}(x)dx$$

where K_μ is the classical McDonald special function.
Recall the definition of the gamma distribution :

$$\gamma(\mu, b)(dx) = \frac{b^\mu}{\Gamma(\mu)} x^{\mu-1} \exp\left\{ -bx\right\} \mathbf{1}_{(0,\infty)}(x)dx, \quad \mu, b > 0.$$

a) The family of GIG distributions is invariant under f_0:

the image of $\mathrm{GIG}(\mu, a, b)$ by f_0 is $\mathrm{GIG}(-\mu, b, a)$.

b) Barndorff-Nielsen and Halgreen (1977) proved:

$$GIG(-\mu, a, b) * \gamma(\mu, \frac{b^2}{2}) = GIG(\mu, a, b), \quad \mu,\ a,\ b > 0 \tag{1}$$

Therefore if $X \sim \mathrm{GIG}(-\mu, a, a)$ and $Y \sim \mu(\lambda, a^2/2)$ are independent r.v.'s then $X + Y \sim \mathrm{GIG}(\mu, a, a)$ and

$$X \overset{(d)}{=} f_0(X + Y). \tag{2}$$

Letac and Seshadri (1983) proved that (2) characterizes GIG distributions of the type $\mathrm{GIG}(-\lambda, a, a)$.

1.2 The so-called **Matsumoto-Yor property** is the following: let X and Y be two independent r.v.'s such that

$$X \sim \mathrm{GIG}(-\mu, a, b), \;\; Y \sim \gamma(\mu, b^2/2), \quad (\mu, a, b > 0).$$

Then

$$U := f_0(X+Y) = \frac{1}{X+Y}, \;\; V := f_0(X) - f_0(X+Y) = \frac{1}{X} - \frac{1}{X+Y}$$

are independent and

$$U \sim \mathrm{GIG}(-\mu, b, a), \;\; V \sim \gamma(\mu, a^2/2).$$

• The case $a = b$ was proved by Matsumoto and Yor (2001) and a nice interpretation of this property via Brownian motion was given by Matsumoto and Yor (2003).

• The case $\mu = -\frac{1}{2}$ of the Matsumoto-Yor property can be retrieved from an independence property established by Barndorff-Nielsen and Koudou (1998) (see Koudou, 2006).

• Letac and Wesolowski (2000) proved that the Matsumoto-Yor property holds for any $\mu, a, b > 0$ and characterizes the GIG distributions.
Consider two independent and non-Dirac positive r.v.'s X and Y such that

$$U := f_0(X + Y), \quad V := f_0(X) - f_0(X + Y)$$

are independent, then there exist $\mu, a, b > 0$ such that

$$X \sim \mathrm{GIG}(-\mu, a, b), \quad Y \sim \gamma(\mu, b^2/2).$$

1.3 The formulation of the Matsumoto-Yor property joined with the Letac and Wesolowski result lead us to determine the triplets :

$$(\mu_X, \mu_Y, \xi)$$

such that

- μ_X, μ_Y are probability measures on $(0, \infty)$,
- $\xi : (0, \infty) \to (0, \infty)$ is bijective and decreasing,
- if X and Y are independent r.v.'s such that

$$X \sim \mu_X \text{ and } Y \sim \mu_Y$$

then the r.v.'s

$$U = \xi(X + Y) \text{ and } V = \xi(X) - \xi(X + Y)$$

are independent.
ξ is called a Letac Wesolowski Matumoto Yor **(LWMY) function**.

Geometric interpretation

Let $I := [X, X+Y]$ a *random* interval : its left-hand X and its length are random and independent.
Let J be the image of I by the map ξ :

$$J := \left[\xi(X+Y),\ \xi(X)\right] = [U, U+V]$$

with

$$U := \xi(X+Y),\ V := \xi(X) - \xi(X+Y).$$

Under which condition is J a random interval ?

2. Characterization of LWMY functions

Let us introduce :

$$g(x) = \ln\left(\frac{1+x}{x}\right), \quad x > 0$$

$$g^{-1}(x) = \frac{1}{e^x - 1}, \quad x > 0,$$

$$f_\delta(x) = \log\left(\frac{e^x + \delta - 1}{e^x - 1}\right), \quad x > 0 \ (\delta > 0).$$

Theorem 2.1

Let $\xi : (0,\infty) \to (0,\infty)$ be decreasing and bijective. Under some additional assumptions, ξ is a LWMY function if and only if, either ξ equals

$$\alpha f_0(x) = \frac{\alpha}{x} \ \text{ or } \ \alpha g(\beta x) \ \text{ or } \ \alpha g^{-1}(\beta x) \ \text{ or } \ \alpha f_\delta(\beta x)$$

for some $\alpha, \beta, \delta > 0$.

Remarks *1) Obviously the case $\xi(x) = 1/x$ corresponds to the Letac Wesolowski and Matsumoto Yor properties.*
2) Associated with a bijective function ξ, consider the transformation $T_\xi : (0,\infty)^2 \to (0,\infty)^2$:

$$(x, y) \mapsto \big(\xi(x+y),\ \xi(x) - \xi(x+y)\big).$$

The transformation T_ξ is one-to-one and if ξ^{-1} is the inverse of ξ, then $(T_\xi)^{-1} = T_{\xi^{-1}}$. Namely :

$$(U, V) = T_\xi(X, Y) = \Big(\xi(X+Y),\ \xi(X) - \xi(X+Y)\Big)$$

if and only if

$$(X, Y) = \Big(\xi^{-1}(U+V),\ \xi^{-1}(U) - \xi^{-1}(U+V)\Big).$$

Therefore ξ is a LWMY function if and only if ξ^{-1} is a LWMY function.

3) The assumptions are

• μ_X *and* μ_Y *have densities of class* C^2

• $f'(x) < 0$ *and*

$$\frac{1}{f'(x)} = \sum_{n \geq 1} a_n x^n, \quad x > 0.$$

3. Distributions related to the function g

We restrict ourselves to

$$\mu_X(dx) = p_X(x)dx, \quad \mu_Y(dx) = p_Y(x)dx.$$

It is supposed that p_X and p_Y are positive and that $\log p_X$ and $\log p_Y$ are locally integrable over $]0, \infty[$.

Consider the case $\xi(x) = g(x) = \ln\left(\dfrac{1+x}{x}\right)$.

Associated with $X, Y > 0$ consider (U, V) defined by

$$\begin{aligned}(U, V) &= \Big(g(X+Y),\ g(X) - g(X+Y)\Big) \\ &= \left(\ln\left(1 + \frac{1}{X+Y}\right), \ln\left(1 + \frac{1}{X}\right) - \ln\left(1 + \frac{1}{X+Y}\right)\right)\end{aligned}$$

Theorem 3.1

Suppose that X and Y are independent. Then, the following are equivalent

1. $U = \ln\left(1 + \frac{1}{X+Y}\right)$ *and* $V = \ln\left(1 + \frac{1}{X}\right) - \ln\left(1 + \frac{1}{X+Y}\right)$ *are independent;*
2. $U' := X + Y$ *and* $V' := \frac{1 + \frac{1}{X+Y}}{1 + \frac{1}{X}}$ *are independent*
3. $X \sim K^{(2)}(a, b, c)$ *and* $Y \sim \gamma(b, c)$ *where*

$$K^{(2)}(a,b,c)(dx) := kx^{a-1}(1+x)^{-a-b}e^{-cx}\mathbf{1}_{(0,\infty)}(x)dx$$

with $a, c > 0$, $b \in \mathbb{R}$ *and* k *being the normalizing constant.*

Morover :

$$U' \sim K^{(2)}(a+b, -b, c), \ \ V' \sim \mathrm{Beta}(a, b)$$

where $\mathrm{Beta}(a,b)(dx) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}\mathbf{1}_{\{0<x<1\}}dx.$

Remark

1. *The densities of U and V are respectively :*

$$p_U(x) = \alpha e^{-(a+b)x}(1-e^{-x})^{-b-1}\exp\left(-c\frac{e^{-x}}{1-e^{-x}}\right)\mathbf{1}_{\{x>0\}}$$

$$p_V(y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}(1-e^{-y})^{b-1}e^{-ay}\mathbf{1}_{\{y>0\}}.$$

2. *We have a similar result starting with U and V independent. Then $X := g^{-1}(U+V) = \dfrac{1}{e^{U+V}-1}$ and*

$$Y := g^{-1}(U) - g^{-1}(U+V) = \frac{1}{e^{U}-1} - \frac{1}{e^{U+V}-1}$$

are independent if and only if ...

Proposition 3.2

We have :

$$K^{(2)}(a,b,c) * \gamma(b,c) = K^{(2)}(a+b,-b,c) \quad (a,b,c>0).$$

4. Distributions associated with the function f_δ

4.1 General results

Recall that

$$f_\delta(x) = \ln\left(\frac{e^x + \delta - 1}{e^x - 1}\right), \ \ x > 0 \ (\delta > 0).$$

Associated with X and Y consider :

$$(U, V) := \Big(f_\delta(X + Y),\ f_\delta(X) - f_\delta(X + Y)\Big).$$

It is more convenient to introduce :

$$X' := e^{-X},\ Y' := e^{-Y},\ U' := e^{-U},\ V' := e^{-V}.$$

Note that X', Y', U' and V' take their values in $]0, 1[$ and :

$$(U', V') = \left(\frac{1 - X'Y'}{1 + (\delta - 1)X'Y'}, \frac{1 - X'}{1 + (\delta - 1)X'}\frac{1 + (\delta - 1)X'Y'}{1 - X'Y'}\right).$$

Let us introduce the following probability measure over $[0, 1]$:

$$\beta_\delta(a, b; c)(dx) = kx^{a-1}(1 - x)^{b-1}(\delta x + 1 - x)^c \mathbf{1}_{(0,1)}(x)dx$$

where $a, b > 0$ and $c \in \mathbb{R}$ and k is the normalization factor.

Theorem 4.1

Suppose that X' and Y' are independent and the log-densities of $p_{X'}$ and $p_{Y'}$ are locally integrable over $]0, \infty[$. Then $U' = \dfrac{1 - X'Y'}{1 + (\delta - 1)X'Y'}$ and $V' = \dfrac{1 - X'}{1 + (\delta - 1)X'} \dfrac{1 + (\delta - 1)X'Y'}{1 - X'Y'}$ are independent if and only if

$$X' \sim \beta_\delta(a + b, \lambda; -\lambda - b) \ \ \textit{and} \ \ Y' \sim \mathrm{Beta}(a, b)$$

for some $a, b, \lambda > 0$.
Moreover,

$$U' \sim \beta_\delta(\lambda + b, a; -a - b), \ \ V' \sim \mathrm{Beta}(\lambda, b),$$

Proposition 4.2

Suppose that

$$X' \sim \beta_\delta(a+b,\lambda;-\lambda-b), \quad \textit{and} \quad Y' \sim \mathrm{Beta}(a,b)$$

where $a, b, \lambda > 0$.

Then the r.v. $U' = \dfrac{1 - X'Y'}{1 + (\delta-1)X'Y'}$ *is* $\beta_\delta(\lambda+b, a; -a-b)$*-distributed.*

4.2 The particular case $\delta = 1$

We have :

$$f_1(x) = \ln\left(\frac{e^x}{e^x - 1}\right), \quad x > 0$$

and $\beta_1(a,b;c) = \mathrm{Beta}(a,b)$.

Theorem 4.3

Let X' and Y' be two independent random variables valued in $(0, 1)$. Then

$$U' = 1 - X'Y', \ V' = \frac{1 - X'}{1 - X'Y'}$$

are independent if and only if there exist $a, b, \lambda > 0$ such that

$$X' \sim \mathrm{Beta}(a + b, \lambda) \text{ and } Y' \sim \mathrm{Beta}(a, b).$$

If one of these conditions holds, then $U' \sim \mathrm{Beta}(\lambda + b, a)$ and $V' \sim \mathrm{Beta}(\lambda, b)$.

4.3 A geometric interpretation

A *multiplicative random interval* is of the type $I' = [X'Y',\ X']$, where X' and Y' are independent and belong to $]0,1[$.

Set

$$\overline{f_\delta}(x) := \frac{1-x}{1+(\delta-1)x}, \quad 0 < x < 1.$$

Let J' be the image of I' by $\overline{f_\delta}$:

$$J' := \left[\overline{f_\delta}(X'),\ \overline{f_\delta}(X'Y')\right] = [U'V', U']$$

where

$$U' := \overline{f_\delta}(X'Y') = \frac{1-X'Y'}{1+(\delta-1)X'Y'}.$$

$$V' = \frac{\overline{f_\delta}(X')}{\overline{f_\delta}(X'Y')} = \frac{1-X'}{1+(\delta-1)X'}\frac{1+(\delta-1)X'Y'}{1-X'Y'}$$

Geometric version of Theorem 4.1

Suppose that X' and Y' are independent and the log-densities of $p_{X'}$ and $p_{Y'}$ are locally integrable over $]0,\infty[$. Then J' is a multiplicative random interval if and only if

$$X' \sim \beta_\delta(a+b,\lambda;-\lambda-b) \ \text{ and } \ Y' \sim \mathrm{Beta}(a,b)$$

for some $a, b, \lambda > 0$.

4.4 A limit procedure

We have :

$$\lim_{\delta \to 0} f_\delta(\delta x) = \lim_{\delta \to 0} \left[\ln \left(\frac{e^{\delta x} + \delta - 1}{e^{\delta x} - 1} \right) \right] = \ln \left(\frac{1 + x}{x} \right) = g(x)$$

Let X_δ and Y_δ be r.v.'s associated with the LWMY function f_δ as in Theorem 4.1 :

$$f_\delta(X_\delta + Y_\delta) \text{ and } f_\delta(X_\delta) - f_\delta(X_\delta + Y_\delta)\Big) \text{ are independent.}$$

This property is equivalent to :

$$f_\delta\Big(\delta \, \big[\frac{X_\delta}{\delta} + \frac{Y_\delta}{\delta}\big] \Big) \text{ and } f_\delta\Big(\delta \, \tfrac{X_\delta}{\delta}\Big) - f_\delta\Big(\delta \, \big[\tfrac{X_\delta}{\delta} + \tfrac{Y_\delta}{\delta}\big] \Big)$$

are independent.

Proposition 4.4

Suppose that X_δ *and* Y_δ *are independent and* $f_\delta(X_\delta + Y_\delta)$ *and* $f_\delta(X_\delta) - f_\delta(X_\delta + Y_\delta)\big)$ *are independent.*

Then

1. $\dfrac{X_\delta}{\delta}$ *(resp.* $\dfrac{Y_\delta}{\delta}$*) converges in distribution, as* $\delta \to 0$ *to* X_0 *(resp.* Y_0*),*
2. X_0 *and* Y_0 *are independent and*
$$X_0 \sim K^{(2)}(a, b, c), \quad Y_0 \sim \gamma(b, c).$$
3. $g(X_0 + Y_0)$ *and* $g(X_0) - g(X_0 + Y_0)$ *are independent.*