Diffusions with jumps

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In the talk we consider a wide class of homogeneous diffusions with jumps. The extreme points of this class are homogeneous diffusion processes and the Poisson processes with variable intensity. The diffusions with jumps have many good properties inherited both from classical diffusion processes and from Poisson ones. This class is closed with respect to a composition with the invertible twice continuously differentiable functions. A special random time change gives us again a diffusion with jumps. A result on transformation of the measure of the process analogous to Girsanov's transformation is valid for this class. There are an effective results for the computation of distributions of certain functionals of diffusions with jumps.

1 Diffusion with jumps

Let $N(t)$, $t \geq 0$, be a Poisson process with the parameter of intensity 1. The process $N(t)$ can be represented in the following form

$$N(t) = \max\Big\{l : \sum_{k=1}^{l} \tau_k \leq t\Big\},$$

where the τ_k, $k = 1, 2, \dots$, are independent exponentially distributed with parameter 1 random variables ($\mathbf{P}(\tau_k \geq t) = e^{-t}$).

Let Y_k, $k = 1, 2, \dots$, be i.i.d. random variables that are independent of the process N. Assume that the Brownian motion $W(t)$, $t \geq 0$, is independent of the variables τ_k and Y_k, $k = 1, 2, \dots$.

The process

$$N_c(t) := \sum_{k=1}^{N(\lambda_1 t)} Y_k$$

is called a *compound Poisson* process with the intensity of jumps λ_1.

This process can be interpreted as a degenerate diffusion with jumps for which the diffusion is a constant process. The most interesting diffusion with jumps is the process

$$J^{(\mu)}(t) := \mu t + \sigma W(t) + \sum_{k=1}^{N(\lambda_1 t)} Y_k, \qquad W(0) = x,$$

where $W(t)$, $t \geq 0$, is a Brownian motion which is independent of the process N and the variables Y_k, $k = 1, 2, \ldots$. The process $J^{(\mu)}$ is called *Brownian motion with linear drift with jumps.* This process is a homogeneous process with independent increments.

One of the main points of a wide class of diffusions with jumps under consideration is that the values of jumps may depend not only on the variables Y_k, $k = 1, 2, \ldots$, but also on the position of the diffusion before a jump. Therefore the value of a jump is defined by a measurable function $\rho(x, y)$, where the first argument is reserved for values of the diffusion before the jump and the second one corresponds to the variables Y_k. It is assumed that the function $\rho(x, y)$ is right-continuous in x uniformly in y and has the left-limits uniformly in y.

The next generalization is connected with the moments of jumps. In usual situation, these moments are the jumping moments of the Poisson process N, i.e., the moments follow each other over the intervals τ_k, $k = 1, 2, \ldots$. It is possible to consider the more general jumping moments which depend on the behavior of the diffusion between the jumps. These moments are the first hitting times of the levels τ_k by some integral functional of the diffusion.

Consider a homogeneous diffusion X. Such a diffusion is the solution of the following stochastic differential equation

$$X(t) = X(s) + \int_s^t \mu(X(u))\,du + \int_s^t \sigma(X(u))\,dW(u) \qquad \text{a.s.} \tag{1}$$

for any $s \le t \le T$, where the functions $\mu(x)$ and $\sigma(x)$ satisfy the Lipschitz condition, $\sigma(x) \neq 0$ for all $x \in \mathbf{R}$ and $X(0) = x$.

For a nonnegative piecewise continuous function h the moment

$$\varkappa_1 := \min\left\{s : \int_0^s h(X(v))\,dv = \tau_1\right\}$$

is the moment inverse to the integral functional of the diffusion X.

A **diffusion with jumps** (denoted J) is defined recurrently as follows. For $0 = \varkappa_0 \le t \le \varkappa_1$, set $J(t) := X(t)$, where X is the solution of (1). For $l = 1, 2, \ldots,$ the process J is the solution of the following stochastic differential equation:

$$J(t) = \rho\big(J(\varkappa_l -), Y_l\big) + \int_{\varkappa_l}^{t} \mu(J(u))\, du + \int_{\varkappa_l}^{t} \sigma(J(u))\, dW(u) \tag{2}$$

on the time interval $\varkappa_l \le t < \varkappa_{l+1}$, where

$$\varkappa_{l+1} := \min\Big\{ s \ge \varkappa_l : \int_{\varkappa_l}^{s} h(J(v))\, dv = \tau_{l+1} \Big\}.$$

Consider the family $X_{s,x}(t)$, $x \in \mathbf{R}$, $t \in [s,T]$, of solutions of the stochastic differential equations

$$X_{s,x}(t) = x + \int_s^t \mu(X_{s,x}(u))\,du + \int_s^t \sigma(X_{s,x}(u))\,dW(u).$$

Then

$$X(t) = X_{s,X(s)}(t), \qquad s \le t \le T, \qquad \text{a.s.}$$

for every fixed s. It is clear that if $\varkappa_l \le t < \varkappa_{l+1}$, then

$$J(t) = X_{\varkappa_l,\rho(J(\varkappa_l-),Y_l)}(t), \tag{3}$$

where

$$\varkappa_{l+1} := \min\Big\{ s \ge \varkappa_l : \int_{\varkappa_l}^s h(X_{\varkappa_l,\rho(J(\varkappa_l-),Y_l)}(v))\,dv = \tau_{l+1} \Big\}.$$

At any moment $\varkappa_l$, the diffusion J is started anew as a usual diffusion X from the point $\rho\big(J\big(\varkappa_l-\big),Y_l\big)$.

The mechanism of appearance of the moments of jumps corresponds to the Poisson process with varying intensity and it consists of the following.

After a jumping moment we consider a sample path of the diffusion. Since a diffusion with jumps starts anew at moments of jumps, it is possible to consider the mechanism at the initial time moment.

We decompose the time interval $[0, t]$ into infinitely small disjoint subintervals $[s, s+ds)$. Assume that under a fixed sample path $X(\cdot)$ on each such a subinterval independently of others the moment of jump can appear with probability $h(X(s))\, ds$. It means that the diffusion X continue its movement without jumps with probability $(1 - h(X(s))\, ds \approx e^{-h(X(s))\, ds}$. Then, for a fixed sample path of the diffusion, the tail probability of the moment of the first jump has the form

$$\mathbf{P}(\varkappa_1 > t | X(\cdot)) \approx \exp\Big(- \int_0^t h(X(v))\, dv \Big) = \mathbf{P}\Big(\int_0^t h(X(v))\, dv \le \tau_1 \Big| X(\cdot) \Big).$$

This exactly corresponds to the conditional distribution of the moment inverse to the integral functional of the diffusion X at the level τ_1.

A diffusion with jumps J is characterized by the following parameters:

coefficients of the diffusion $\mu(x)$ and $\sigma(x)$, the function of jumps $\rho(x,y)$, the distribution $F(x)$, $x \in \mathbf{R}$, of the random variables Y_k, $k = 1, 2, \dots$, and the intensity function $h(x)$, which specifies the time intervals between jumps.

The process $C(t) = \max\{l : \varkappa_l \le t\}$, $t \ge 0$, similarly to the Poisson process, counts the number of jumps performed by the diffusion J up to time t, and $dC(t)$ equals one if $\varkappa_l$ belongs to the interval $[t, t+dt)$ for some $l = 1, 2, \dots$, and equals zero otherwise.

It is easy to understand that *counting process* can be represented as follows $C(t) = N(A(t))$, where $A(t) := \int_0^t h(J(v))\,dv$.

The differential form of the equation (2) is the following

$$dJ(t) = \mu(J(t))\,dt + \sigma(J(t))\,dW(t) + \big(\rho\big(J(t-), Y_{C(t)}\big) - J(t-)\big)dC(t). \quad (4)$$

(Gihman, Skorohod (1968) considered in nonhomogeneous case such a SDE for $h \equiv 1$ and for more general Poisson random measure $\nu(dt, dx)$. Gihman, Skorohod (1975) derived a more general equation).

Let $b(x)$, $x \in \mathbf{R}$, be a twice continuously differentiable function. The following generalization of **Itô's formula** holds

$$db(J(t)) = b'(J(t))(\mu(J(t))\,dt + \sigma(J(t))\,dW(t)) + \frac{1}{2}\sigma^2(J(t))b''(J(t))\,dt$$

$$+\big(b(\rho(J(t-), Y_{C(t)})) - b(J(t-))\big)\,dC(t). \tag{5}$$

The result of applying the expectation to the Itô formula is of key importance. For the diffusion with jumps we deduce that

$$d\mathbf{E}_x b(J(t)) = \mathbf{E}_x\{b'(J(t))\mu(J(t))\}\,dt + \frac{1}{2}\mathbf{E}_x\{\sigma^2(J(t))b''(J(t))\}\,dt$$

$$+\mathbf{E}_x\big\{h(J(t))\big(b(\rho(J(t), Y_1)) - b(J(t))\big)\big\}\,dt. \tag{6}$$

Compositions of diffusions with jumps with invertible functions. Let $b(x)$ be a twice continuously differentiable function with the inverse function $b^{(-1)}(x)$, i.e., $b^{(-1)}(b(x)) = x$. Then $\widetilde{J}(t) := b(J(t))$ is again the diffusion with jumps. Its parameters are the following:

$$\tilde{\rho}(x, y) := b(\rho(b^{(-1)}(x), y)), \qquad \tilde{\sigma}(x) := b'(b^{(-1)}(x))\sigma(b^{(-1)}(x)),$$

$$\tilde{\mu}(x) := b'(b^{(-1)}(x))\mu(b^{(-1)}(x)) + \frac{1}{2}b''(b^{(-1)}(x))\sigma^2(b^{(-1)}(x)).$$

Therefore the class of diffusions with jumps is closed with respect to the compositions with invertible twice continuously differentiable functions.

2 Examples of diffusions with jumps

2.1 Geometric Brownian motion with jumps

Assume that the Brownian motion W, the Poisson process N, and the variables $\{Y_k\}_{k=1}^{\infty}$ are independent. Set $\nu := \mu - \sigma^2/2$ and

$$J^{(\nu)}(t) := (\mu - \sigma^2/2)t + \sigma W(t) + \sum_{k=1}^{N(t)} Y_k, \qquad W(0) = 0. \tag{7}$$

Let $b(x) = e^x$. By Itô's formula (5) the process $V(t) = e^{J^{(\nu)}(t)}$ is the solution of the following linear equation

$$dV(t) = \mu V(t)\,dt + \sigma V(t)\,dW(t) + V(t-)\big(e^{Y_{N(t)}} - 1\big)\,dN(t), \qquad V(0) = 1, \tag{8}$$

since $\tilde{\rho}(x, y) = \exp(\log x + y) = xe^y$.

It is natural to call the process V a *geometric* (*exponential*) *Brownian motion with jumps* by analogy with a diffusion without jumps. These processes are often used in different models connected with financial mathematics.

2.2 Bessel process with jumps

Next example arises by analogy with the Bessel process, which is the radial part of the multidimensional Brownian motion with independent coordinates.

Let $\{W_l(s), s \geq 0\}$, $l = 1, 2, \ldots, n$, be a family of independent Brownian motions. The process $R^{(n)}$ defined by the formula

$$R^{(n)}(t) := \sqrt{W_1^2(t) + W_2^2(t) + \cdots + W_n^2(t)}, \quad t \geq 0,$$

is called an *n-dimensional Bessel process* or a *Bessel process of the order* $\frac{n}{2} - 1$.

Let diffusions $\{J_l(s), s \geq 0\}$, $l = 1, 2, \ldots, n$, be defined by the following equations

$$dJ_l(t) = dW_l(t) + \left(\sqrt{\alpha J_l^2(t-) + Y_{N(t)}^{(l)}} - J_l(t-)\right) dN(t),$$

where $(Y_k^{(1)}, Y_k^{(2)}, \ldots, Y_k^{(n)})$, $k = 1, 2, \ldots$, are i.i.d. random vectors with nonnegative coordinates and α is an arbitrary nonnegative constant. The choice of the jump function $\rho(x, y) = \sqrt{\alpha x^2 + y}$ is preset by our wish to obtain the following result.

Proposition 1 *The radial part of the multidimensional diffusion with jumps* $\vec{J}(t) = (J_1(t), J_2(t), \ldots, J_n(t))$ *is a diffusion with jumps.*

Indeed. Set $Z_n(t) := \sqrt{J_1^2(t) + J_2^2(t) + \cdots + J_n^2(t)}, \quad t \geq 0.$

Then Z_n is the diffusion with jumps defined by the equation

$$dZ_n(t) = \frac{n-1}{2Z_n(t)}\,dt + dW(t) + \left(\sqrt{\alpha Z_n^2(t-) + S_{N(t)}} - Z_n(t-)\right) dN(t), \quad (9)$$

where $S_k := \sum\limits_{l=1}^{n} Y_k^{(l)}$.

To prove this, we apply Itô's formula (5) (where $b(x) = x^2$) to each term of the process $L_n(t) := Z_n^2(t)$. In this case,

$$dL_n(t) = 2\sum_{l=1}^{n} J_l(t)\,dW_l(t) + n\,dt + \sum_{l=1}^{n} \big((\alpha - 1)J_l^2(t-) + Y_{N(t)}^{(l)}\big)dN(t).$$

Again applying Itô's formula (5) (where $b(x) = \sqrt{x}$) we obtain (9).

3 Distribution of integral functionals and of infimum and supremum functionals

Let $\mu(x)$ and $\sigma(x)$, $x \in \mathbf{R}$, be continuously differentiable functions, satisfying the following conditions:

$$|\mu(x)| + \sigma(x) \le C(1 + |x|), \qquad \text{for all} \quad x \in \mathbf{R},$$

and $\inf_{x\in\mathbf{R}} \sigma(x) > 0$. Assume also that the derivative $\left(\frac{\mu(x)}{\sigma^2(x)}\right)'$ is bounded.

Consider the method of computing the distribution of the functional

$$A(t) := \int_0^t f(J(s))\, ds, \qquad f \ge 0,$$

and of the infimum and supremum functionals $\inf_{0\le s\le t} J(s)$, $\sup_{0\le s\le t} J(s)$.

Let τ be the random moment that is independent of the process J and exponentially distributed with parameter $\lambda > 0$.

The random time τ corresponds to the Laplace transform with respect to the fixed time t.

We denote by $\mathbf{P}_x$ and $\mathbf{E}_x$ the probability and expectation with respect to the process J under the condition $J(0) = x$.

Assume that $\inf\limits_{x\in\mathbf{R}} h(x) > 0$. This guarantee that $\varkappa_1 < \infty$ a.s.

Theorem 1 *Let $\varPhi(x)$, $f(x)$, and $h(x)$, $x \in [a,b]$, be piecewise continuous functions. Assume that $f \geq 0$, $q(x,y)$, $(x,y) \in [a,b] \times \mathbf{R}$, is a non-negative function, which is right-continuous in x uniformly in y, and has the left-limits uniformly in y. Then the function*

$$Q(x) := \mathbf{E}_x \Big\{ \varPhi(J(\tau)) \exp\Big(-\int_0^\tau f(J(s))\, ds - \int_{[0,\tau]} q\big(J(s-), Y_{C(s)}\big)\, dC(s) \Big)$$

$$1_{\left\{a \leq \inf\limits_{0\leq s\leq\tau} J(s),\ \sup\limits_{0\leq s\leq\tau} J(s) \leq b\right\}} \Big\}$$

is the unique bounded solution of the equation

$$Q(x) = M(x) + \int_{-\infty}^{\infty} G_z(x)\, \mathbf{E}\{e^{-q(z,Y_1)} Q(\rho(z,Y_1))\}\, dz, \tag{10}$$

where $M(x)$, $x \in (a,b)$, *is the unique solution of the problem*

$$\frac{\sigma^2(x)}{2} M''(x) + \mu(x) M'(x) - (\lambda + h(x) + f(x)) M(x) = -\lambda \Phi(x), \tag{11}$$

$$M(a) = 0, \qquad M(b) = 0, \tag{12}$$

and $G_x(z)$, $z \in (a,b) \setminus \{x\}$, *is the unique solution of the problem*

$$\frac{\sigma^2(x)}{2} G''(x) + \mu(x) G'(x) - (\lambda + h(x) + f(x)) G(x) = 0, \tag{13}$$

$$G'(z+0) - G'(z-0) = -2h(z-)/\sigma^2(z), \tag{14}$$

$$G(a) = 0, \qquad G(b) = 0. \tag{15}$$

It is set $M(x) = 0$, $G_x(z) = 0$ *if* $x, z \notin (a,b)$.

The probabilistic expressions for the functions $M(x)$ and $G_x(z)$, $x \in \mathbf{R}$, $z \in \mathbf{R}$, are the following:

$$M(x) = \mathbf{E}_x\Big\{\varPhi(X(\tau)) \exp\Big(- \int_0^\tau (h(X(s)) + f(X(s)))\, ds\Big)$$

$$1_{\big\{a \le \inf\limits_{0 \le s \le \tau} X(s),\ \sup\limits_{0 \le s \le \tau} X(s) \le b\big\}}\Big\},$$

$$G_z(x) = \frac{d}{dz}\mathbf{E}_x\Big\{ \exp\Big(- \int_0^{\varkappa_1} (\lambda + f(X(s)))\, ds\Big)$$

$$1_{\big\{a \le \inf\limits_{0 \le s \le \varkappa_1} X(s),\ \sup\limits_{0 \le s \le \varkappa_1} X(s) \le b,\, X(\varkappa_1) < z\big\}}\Big\}.$$

4 Random time change

Let a diffusion with jumps $J(t)$, $t \geq 0$, be specified by parameters $\big(\mu(x)$, $\sigma(x)$, $\rho(x,y)$, $\{Y_k\}$, $h(x)\big)$, i.e. this diffusion is the solution of the stochastic differential equation

$$dJ(t) = \mu(J(t))\,dt + \sigma(J(t))\,dW(t) + \big(\rho\big(J(t-), Y_{C(t)}\big) - J(t-)\big)dC(t), \;\; J(0) = x,$$

with $\sigma(x) > 0$ and space state $\mathbf{R}$. Let $g(x)$, $x \in \mathbf{R}$, be a twice continuously differentiable function with $g'(x) \neq 0$ (thus this function has the inverse $g^{-1}(x)$, $x \in g(\mathbf{R})$). Consider the integral functional

$$A_t := \int_0^t \big(g'(J(s))\sigma(J(s))\big)^2\,ds, \qquad t \in [0, \infty),$$

as the function of the upper limit. Assume that $A_\infty = \infty$. Define the inverse process:

$$a_t := \min\big\{s : A_s = t\big\}, \quad t \in [0, \infty).$$

Since A_t is the strictly increasing function, $\alpha_0 = 0$.

Theorem 2 *The process $\widetilde{J}(t)$ defined by the formula*

$$\widetilde{J}(t) = g\big(J(a_t)\big), \quad t \in [0, \infty), \tag{16}$$

is the diffusion with jumps specified by parameters $\big(\widetilde{\mu}(x), 1, \widetilde{\rho}(x,y), \{Y_k\}, \widetilde{h}(x)\big)$, where

$$\widetilde{\mu}(x) = D(g^{-1}(x)), \qquad \frac{g''(x)}{2(g'(x))^2} + \frac{\mu(x)}{g'(x)\sigma^2(x)},$$

$$\widetilde{\rho}(x,y) = g(\rho(g^{-1}(x), y)), \qquad \widetilde{h}(x) = \frac{h(g^{-1}(x))}{(g'(g^{-1}(x))\sigma(g^{-1}(x)))^2},$$

i.e. $\widetilde{J}(t)$ is the solution of the stochastic differential equation

$$d\widetilde{J}(t) = d\widetilde{W}(t) + \widetilde{\mu}(\widetilde{J}(t))\,dt + \Big(\widetilde{\rho}\big(\widetilde{J}(t-), Y_{\widetilde{C}(t)}\big) - \widetilde{J}(t-)\Big)d\widetilde{C}(t), \quad \widetilde{J}(0) = g(x),$$

where $\widetilde{W}(t)$ is a Brownian motion and

$$\widetilde{C}(t) = N\Big(\int_0^t \widetilde{h}(\widetilde{J}(v))\,dv\Big).$$

5 Transformation of the measure

Consider two homogeneous diffusions:

$$dX_l(t) = \sigma(X_l(t))dW(t) + \mu_l(X_l(t))dt, \qquad X_l(0) = x, \qquad l = 1, 2,$$

and two sequences of i.i.d. random variables $Y_k^{(l)}$ with absolutely continuous distributions $\frac{dF_2(y)}{dF_1(y)} = p(y)$. Let J_l, $l = 1, 2$, be two diffusions with jumps defined by X_l, $Y_k^{(l)}$, intensity functions $h_l(x)$, $l = 1, 2$, and having the same function $\rho(x, y)$. Denote by C_l the process which counts the number of jumps performed by the diffusion J_l up to time t.

Let $D[0, t]$ be the space of functions on $[0, t]$ without discontinuities of the second type. The result about the transformation of the measure of the diffusions with jumps can be formulated as follows. Under some conditions, for each t and any bounded continuous function $\wp(Z(s), 0 \le s \le t)$ on $D[0, t]$

$$\mathbf{E}\wp(J_2(s), 0 \le s \le t) = \mathbf{E}\big\{\wp(J_1(s), 0 \le s \le t)\Theta(t)\big\}, \tag{17}$$

where

$$\Theta(t) = \prod_{k=1}^{C_1(t)} p(Y_k^{(1)}) \exp\Big(-\int_0^t \big(h_2(J_1(s)) - h_1(J_1(s))\big) ds + \int_{[0,t]} \ln \frac{h_2(J_1(s-))}{h_1(J_1(s-))} dC_1(s)\Big)$$

$$\times \exp\Big(\int_0^t \frac{\mu_2(J_1(s)) - \mu_1(J_1(s))}{\sigma(J_1(s))}\, dW(s) - \int_0^t \frac{(\mu_2(J_1(s)) - \mu_1(J_1(s)))^2}{2\sigma^2(J_1(s))}\, ds\Big).$$

Remark For $\rho(x, y) = x$ the processes J_l have no jumps and they are homogeneous diffusions. In this case formula (17) is transformed into the classical one.

One can check this in the following way. Since in this case the Poisson process N independent of the diffusion J_1 and the process $C_1(t)$ has the intensity function $h_1(J_1(t))$, then the following equalities hold

$$\mathbf{E}_N \exp\Big(\int_{[0,t]} \ln \frac{h_2(J_1(s-))}{h_1(J_1(s-))} dC_1(s)\Big)$$

$$= \exp\Big(\int_0^t \Big(\exp\Big(\ln \frac{h_2(J_1(s))}{h_1(J_1(s))}\Big) - 1\Big) h_1(J_1(s))\, ds\Big)$$

$$= \exp\Big(\int_0^t \big(h_2(J_1(s)) - h_1(J_1(s))\big)\, ds\Big),$$

where subindex N denotes that the expectation is computed only with respect to the process N. Besides that $\mathbf{E}p(Y_k^1) = 1$. Now, using an independence of the processes J_1, N, Y_k^1, $k = 1, 2, \ldots$, and applying Fubini's theorem, one can check that (17) is transformed into the classical one.

Using the formula of stochastic differentiation, it is possible to rewrite the derivative $\Theta(t)$ without the stochastic integral.

Set

$$\delta(x) := \frac{1}{\sigma^2(x)}(\mu_2(x) - \mu_1(x)) \quad \text{and} \quad \Delta(x) := \int_0^x \delta(y)\, dy.$$

Assume that δ is a continuously differentiable function. Then

$$\Theta(t) = e^{\Delta(J_1(t)) - \Delta(J_1(0))} \exp\Big(- \int_0^t \big(h_2(J_1(s)) - h_1(J_1(s))\big)\, ds\Big)$$

$$\times \exp\Big(- \int_0^t \frac{\mu_2^2(J_1(s)) - \mu_1^2(J_1(s))}{2\sigma^2(J_1(s))}\, ds - \int_0^t \frac{\sigma^2(J_1(s))}{2} \delta'(J_1(s))\, ds$$

$$
-\int_{[0,t]} \Big(\Delta\big(\rho\big(J_1(s-), Y^{(1)}_{C_1(s)}\big)\big) - \Delta(J_1(s-)) - \ln \frac{h_2(J_1(s-))}{h_1(J_1(s-))}\Big)\, dC_1(s)\Big)
$$

$$
\times \prod_{k=1}^{C_1(t)} p(Y^{(1)}_k).
$$

The statement given by formula (17) is equivalent to the following one: for an arbitrary $\lambda > 0$ and for a random moment τ that is exponentially distributed with parameter λ and independent of the processes J_l, $l = 1, 2$, the following equality holds

$$
\mathbf{E}_x \wp(J_2(s), 0 \le s \le \tau) = \mathbf{E}_x \big\{ \wp(J_1(s), 0 \le s \le \tau)\Theta(\tau) \big\}. \tag{18}
$$

References

[1] Gihman, I. I., Skorohod A. V., *Stochastic Differential Equations*. Naukova Dumka, Kiev, 1968.

[2] Gihman, I. I., Skorohod A. V., *Theory of Stochastic Processes* III. Nauka, M., 1975.

[3] Merton, R. C., *Option pricing when the underlying stock returns are discontinuous*. — J. Financial Economics **3** (1976), 123–144.

[4] Mordecki, E., *Ruin probabilities for Lévy processes with mixed exponential negative jumps*. — Theory probab. appl. **48**, No. 1 (2003), 188–194.

[5] Borodin, A. N., *Distributions of functionals of diffusions with jumps*. — Zap. nauchn. semin. POMI **339** (2006), 15–36. (Engl. transl. J. Math. Sc.)

[6] Borodin, A. N., *Transformations of diffusions with jumps*. — Zap. nauchn. semin. POMI **351** (2007), 79–100. (Engl. transl. J. Math. Sc.)