Simple Arbitrage

Motivated by and partly based on a joint work with T. Sottinen and E. Valkeila

Christian Bender

Saarland University

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Problem Setting

- Financial market with two assets (for simplicity) on $[0, T]$: constant bond $B_t = 1$, a stock X_t (adapted process with continuous paths).
- Simple strategy (number of risky shares held by the investor):

$$\Phi_t = \phi_0 \mathbf{1}_{\{0\}}(t) + \sum_{j=0}^{n-1} \phi_j \mathbf{1}_{(\tau_j, \tau_{j+1}]}$$

where the τ_j's are a finite number of ordered stopping times with values in $[0, T]$ and the ϕ_j's are $\mathcal{F}_{\tau_j}$-measurable random variables.

Corresponding wealth process with zero initial endowment:

$$V_t(\Phi) = \sum_{j=0}^{n-1} \Phi_{\tau_{j+1}} (X_{t \wedge \tau_{j+1}} - X_{t \wedge \tau_j})$$

Problem Setting

- A simple arbitrage is a simple strategy which is an arbitrage:

$$V_T(\Phi) \geq 0 \ P\text{-a.s.}, \quad P(\{V_T(\Phi) > 0\}) > 0$$

- Questions:

  How to characterize absence of simple arbitrage?

  What kind of easy sufficient conditions are available for absence of arbitrage (for large subclasses of) simple strategies?

- Motivation: How about simple arbitrage for models driven by fractional Brownian motion?

Conditional full support

- A continuous process S has conditional full support (CFS) if, for every time $0 \leq t \leq T$

$$\text{supp } P(S \in \cdot \,|\mathcal{F}_t^S) = C_{S_t}([t, T]) \quad \text{a.s.},$$

  where $\mathcal{F}^S$ denotes the augmented filtration generated by S.
- Examples: Fractional Brownian motion B^H satisfies conditional full support for every choice of the Hurst parameter $0 < H < 1$ and so does mixed fractional Brownian motion $B^H + W$, where W is a Brownian motion independent of W. Cf. Cherny (2008) or Gasbarra, Sottinen, van Zanten (2011).

A criterion based on conditional full support

Theorem (B./Sottinen/Valkeila, 2011)

Suppose $\log(X_t)$ *or* X_t *satisfies (CFS). If the simple strategy* Φ *belongs to the Cheridito class, i.e.* $\tau_{j+1} \geq \tau_j + h$ *for some constant* $h > 0$ *on* $\{\tau_{j+1} > \tau_j\}$*, then* Φ *is not a simple arbitrage for* $(X_t, \mathcal{F}_t^X)$.

Remark: The delay between two stopping times can be localized in a way that e.g. stopping times of the form

$$\tau_{j+1} = \inf\{t > \tau_j;\ X_t - X_{\tau_j} \geq b_t^j\} \wedge T$$

for positive continuous functions b^j are covered, and the above theorem still holds.

A criterion based on conditional full support

Example

Suppose $X_t = \exp\{W_t + t^\alpha\}$ for some $0 < \alpha < 1/2$. Then clearly, $\log(X_t)$ satisfies (CFS). Define the stopping time

$$\tau := \inf\{t > 0;\ \log(X_t) < 0\} > 0$$

by the law of the iterated logarithm. Then, $\Phi_t = \mathbf{1}_{(0,\tau\wedge 1/N]}$ is a simple arbitrage for sufficiently large N.

Note that this arbitrage is 0-admissible, i.e.

$$V_t(\Phi) \geq 0 \quad \text{a.s.}$$

No obvious arbitrage

- In the spirit of Guasoni, Rasonyi, Schachermayer (2010):

  $(X_t, \mathcal{F}_t)$ has no obvious arbitrage (NOA), if for all stopping times σ with $P(\{\sigma < T\}) > 0$ and every $\epsilon > 0$

  $$P(\{\sigma < T\} \cap \{\sup_{t\in[\sigma,T]} X_t < X_\sigma + \epsilon\}) > 0$$

  and

  $$P(\{\sigma < T\} \cap \{\inf_{t\in[\sigma,T]} X_t > X_\sigma - \epsilon\}) > 0$$

- If (NOA) is violated, e.g. there are σ, ϵ such that

  $$P(\{\sup_{t\in[\sigma,T]} X_t \geq X_\sigma + \epsilon\} | \{\sigma < T\}) = 1,$$

  then a simple arbitrage can be obtained by buying one share at time σ and selling it, once the stock price has increased by ϵ.

No obvious arbitrage

- If X has (CFS), then $(X_t, \mathcal{F}_t^X)$ satisfies (NOA), because the (CFS)-property extends automatically to stopping times, see Guasoni, Rasonyi, Schachermayer (2008).

Lemma

$(X_t, \mathcal{F}_t)$ has (NOA) $\Rightarrow$ Every simple arbitrage for $(X_t, \mathcal{F}_t)$ is 0-admissible.

Idea: Suppose a simple arbitrage is not 0-admissible. Then its wealth process drops with positive probability below some level δ. Buying this strategy at such a time, the wealth process must increase by δ (because it is nonnegative at time T). This gives rise to an obvious arbitrage.

Two-way crossing

- Question: How to exclude 0-admissible simple arbitrage?
- Suppose $\sigma \leq T$ is a stopping time and

$$\sigma_{\pm} = \inf\{t \geq \sigma, \quad \pm(X_t - X_\sigma) > 0\} \wedge T.$$

  $(X_t, \mathcal{F}_t)$ satisfies two-way crossing (TWC), if $\sigma_+ = \sigma_-$ a.s. for any stopping time $\sigma \leq T$.
- (TWC) is a condition on the fine structure of the paths. Whenever the stock price moves from X_σ, it will cross the level X_σ infinitely often in time intervals of length ϵ for every $\epsilon > 0$.

Lemma

$(X_t, \mathcal{F}_t)$ satifies (TWC) $\Leftrightarrow$ $(X_t, \mathcal{F}_t)$ is free of 0-admissible simple arbitrage.

A characterization of simple arbitrage

Theorem

The following assertion are equivalent:

(i) $(X_t, \mathcal{F}_t)$ *has no simple arbitrage.*

(ii) $(X_t, \mathcal{F}_t)$ *satisfies (NOA) and (TWC).*

A sufficient condition for (TWC)

Theorem

Suppose $X_t = M_t + Y_t$*, where* M *is continuous* $(\mathcal{F}_t)$*-local martingale and* Y_t *is an* $(\mathcal{F}_t)$*-adapted process. We assume:*

1) There is a strictly positive random variable ϵ *such that for every* $0 \le s \le t \le T$

$$\langle M \rangle_t - \langle M \rangle_s \ge \epsilon(t-s).$$

2) Y *is* $1/2$*-Hölder continuous, i.e. there is a positive random variable* C *such that for every* $0 \le s \le t \le T$

$$|Y_t - Y_s| \le C|t-s|^{1/2}.$$

Then, $(X_t, \mathcal{F}_t)$ *satisfies (TWC).*

Example 1: Mixed fractional Black-Scholes model

Suppose

$$X_t = x_0 \exp\{\sigma B_t^H + \eta W_t + a(t)\}$$

where $\sigma, \eta > 0$, B^H is a fractional Brownian motion with Hurst parameter $H > 1/2$, W is a Brownian motion independent of B^H and $a(t)$ is a deterministic $1/2$-Hölder continuous function, e.g. $a(t) = \mu t - H\sigma^2 t^{2H} - 0.5\eta^2 t$.

Then,

$$\log(X_t) = M_t + Y_t; \quad M_t = \eta W_t, \quad Y_t = \log(x_0) + a(t) + \sigma B_t^H.$$

$(\log(X_t), \mathcal{F}_t^X)$ satisfies (NOA) (because it has conditional full support) and (TWC) (because Y is $1/2$-Hölder continuous).

Consequently $(\log(X_t), \mathcal{F}_t^X)$ is free of simple arbitrage and so is $(X_t, \mathcal{F}_t^X)$.

Example 2: Multi-asset mixed Black-Scholes model

Theorem

Suppose $(W_t, \mathcal{F}_t)$ is an N-dimensional Brownian motion and Z_t is a D-dimensional $\mathcal{F}_t$-adapted process independent of W, which is α-Hölder continuous for some $\alpha > 1/2$. Further assume that the matrix $\sigma\sigma^*$ is strictly positive definite, where $\sigma = (\sigma_{d,\nu})_{d=1,\dots,D,\ \nu=1,\dots,N}$. Define D stocks by

$$X_t^d = x_0^d \exp\left\{\sum_{\nu=1}^{N} \sigma_{d,\nu} W_t^\nu + Z_t^d\right\}$$

with initial values $x_0^d > 0$ for $d = 1, \dots, D$. Then, the D-dimensional mixed Black-Scholes model $(X_t, \mathcal{F}_t)$ with $X_t = (X_t^1, \dots, X_t^D)$ is free of simple arbitrage on $[0, T]$.

More examples...

One can also treat:

- 1-dimensional mixed stochastic volatility models (e.g. mixed Heston models)
- 1-dimensional mixed local volatility models.

Then, (CFS) for the log-prices can be derived from a result by Pakkanen (2010) under suitable conditions and (TWC) follows from the above sufficient condition under suitable conditions.

A question

- Does fractional Brownian motion satisfy (TWC)?

References

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Thank you for your attention

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