On Dufresne's perpetuity, translated and reflected

Paavo Salminen
Åbo Akademi,
Mathematical Department,
FIN-20500 Åbo, Finland,
email: phsalmin@abo.fi

Marc Yor
Université Pierre et Marie Curie,
Laboratoire de Probabilités,
4, Place Jussieu, Case 188
F-75252 Paris Cedex 05, France

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Abstract

Let $B^{(\mu)}$ denote a Brownian motion with drift μ. In this paper we study two perpetual integral functionals of $B^{(\mu)}$. The first one, introduced and investigated by Dufresne in [5], is

$$\int_0^\infty \exp(2\,B_s^{(\mu)})\,ds, \quad \mu < 0.$$

It is known that this functional is identical in law with the first hitting time of 0 for a Bessel process with index μ. In particular, we analyze the following reflected (or one-sided) variants of Dufresne's functional

$$\int_0^\infty \exp(2\,B_s^{(\mu)})\,\mathbf{1}_{\{B_s^{(\mu)}>0\}}\,ds,$$

and

$$\int_0^\infty \exp(2\,B_s^{(\mu)})\,\mathbf{1}_{\{B_s^{(\mu)}<0\}}\,ds.$$

These functionals can also be connected to hitting times. Our second functional, which we call Dufresne's translated functional, is

$$D_\nu := \int_0^\infty (c + \exp(B_s^{(\nu)}))^{-2}\,ds,$$

where c and ν are positive. This functional has all its moments finite, in contrast to Dufresne's functional which has only some finite moments. We compute explicitly the Laplace transform of D_ν in the case $\nu = 1/2$ (other parameter values do not seem to allow explicit solutions) and connect this variable, as well as its reflected variants, to hitting times.

Keywords: Geometric Brownian motion, Brownian motion with drift, Bessel process, Lamperti's transformation, local time, hitting times, occupation times.

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1 Introduction

During the last decade, much interest has been devoted to geometric Brownian motion $G^{(\mu)} = \{G_t^{(\mu)} : t \geq 0\}$ where

$$G_t^{(\mu)} := \exp(B_t^{(\mu)}) := \exp(B_t + \mu t)$$

with B a Brownian motion and μ a real number. This is in part due to the fact that $G^{(\mu)}$ appears as the stock price process in the Black-Scholes model. Unlike for Brownian motion, the quadratic variation of $G^{(\mu)}$ is not deterministic but given by

$$\langle G^{(\mu)} \rangle_t := \int_0^t (G_s^{(\mu)})^2 \, ds, \quad t \geq 0. \tag{1}$$

As a matter of consequence, interest has been focused on this functional, in particular, in relation with Asian options. Recall that the payoff of an Asian option with strike price K and fixed maturity T is given by

$$\left(\frac{1}{T} \langle G^{(\mu)} \rangle_T - K \right)^+.$$

A number of results about the pair $(\, G^{(\mu)} \,,\ \langle G^{(\mu)} \rangle \,)$ can be deduced from Lamperti's time change relationship (see, e.g., the papers in the monograph [24]: no. 5 by Geman and Yor and no. 1 by Yor) given by

$$G_t^{(\mu)} = R^{(\mu)}_{\langle G^{(\mu)} \rangle_t}, \quad t \geq 0, \tag{2}$$

where $R^{(\mu)} = \{R_t^{(\mu)} : t \geq 0\}$ is a Bessel process with index μ (or "dimension" $\delta = 2(1+\mu)$), i.e., $R^{(\mu)}$ is a diffusion on $\mathbf{R}_+$ associated to the generator

$$\frac{1}{2}\frac{d^2}{dx^2} + \frac{2\mu+1}{2x}\frac{d}{dx}, \quad x > 0.$$

Here is a first application of Lamperti's transformation: in the case $\mu < 0$, we have $G_t^{(\mu)} \to 0$ a.s. as $t \to \infty$, and it follows from (2) that

$$\langle G^{(\mu)} \rangle_\infty = \inf\{u : R_u^{(\mu)} = 0\}. \tag{3}$$

Consequently, we have the following result for $\mu < 0$ due to Dufresne [5]

$$\langle G^{(\mu)} \rangle_\infty \stackrel{\text{(d)}}{=} \frac{1}{2\gamma_\nu}, \tag{4}$$

where γ_ν is a gamma-distributed random variable with parameter $\nu = -\mu > 0$ and $\stackrel{\text{(d)}}{=}$ reads "is identical in law with". The identity (4) has been recovered in the paper no. 1 by Yor (in [24]), where a relationship with Getoor's study of last passage times for Bessel processes [6] is being used.

In this paper, $\langle G^{(\mu)} \rangle_\infty$ is called *Dufresne's perpetuity*, and, more generally, $\{\langle G^{(\mu)} \rangle_t : t \geq 0\}$ *Dufresne's functional* or *process*.

Dufresne's process plays a central rôle in the studies of the following exponential type processes associated with $G^{(\mu)}$:

$$\left\{ (G_t^{(\mu)})^{-1} \int_0^t G_s^{(\mu)} \, ds : t \geq 0 \right\}$$

and

$$\left\{ (G_t^{(\mu)})^{-1} \int_0^t (G_s^{(\mu)})^2 \, ds : t \geq 0 \right\}.$$

These are, in fact, $\mathbf{R}_+$-valued diffusions, and may be considered as, respectively, analogues of the celebrated Lévy's and Pitman's theorems concerning reflected Brownian motion and BES(3) process. We refer to the recent works by the second author, jointly with H. Matsumoto, see [13], [14], and [15].

We present now shortly the contents of the paper.

In Section 2, the functionals (with $\mu > 0$)

$$D^{(\mu,\pm)} := \int_0^\infty \exp(-2\,B_s^{(\mu)})\,\mathbf{1}_{\{B_s^{(\mu)} \in \mathbf{R}_\pm\}}\, ds, \tag{5}$$

which we call *Dufresne's reflected perpetuities*, are considered. Comparing the definitions in (1) and (5) we have, obviously

$$\langle G^{(-\mu)} \rangle_\infty \quad \stackrel{\text{(d)}}{=} \quad D^{(\mu,+)} + D^{(\mu,-)}.$$

Using again Lamperti's time change (2) together with some results from Pitman and Yor [16] it is relatively easy to obtain the joint Laplace transform of $D^{(\mu,+)}$ and $D^{(\mu,-)}$, as presented in Theorem 2.2. From this result it may be deduced that $D^{(\mu,+)}$ and $D^{(\mu,-)}$ are identical in law with first hitting times for some diffusions. In particular, we discuss the special case $\mu = 1/2$ where the descriptions take a very appealing form.

It follows from (4) that Dufresne's perpetuity does not have all its moments finite; indeed, for $\mu > 0$

$$\mathbf{E}\Big(\langle G^{(-\mu)} \rangle_\infty^m \Big) < \infty$$

if and only if $m < \mu$. This fact may be undesirable from a financial point of view, and, hence, motivated us to introduce, what we call *Dufresne's translated perpetuity*,

$$\widehat{D}_c^{(\mu)} := \int_0^\infty (c + \exp(B_s^{(\mu)}))^{-2}\, ds, \tag{6}$$

and, further, *Dufresne's translated and reflected perpetuities*

$$\widehat{D}_c^{(\mu,\pm)} := \int_0^\infty \frac{\mathbf{1}_{\{B_s^{(\mu)} \in \mathbf{R}_\pm\}}}{(c + \exp(B_s^{(\mu)}))^2}\, ds. \tag{7}$$

It follows from the general discussion in Salminen and Yor [20] that, for $c > 0$, the functional $\widehat{D}_c^{(\mu)}$ admits some exponential moments and is, in this respect, "much smaller" than Dufresne's perpetuity, $\widehat{D}_0^{(\mu)} = \langle G^{(-\mu)} \rangle_\infty$

On the other hand, we have not been able to compute an explicit formula for the Laplace transform of $\widehat{D}_c^{(\mu)}$, and, a fortiori, for $\widehat{D}_c^{(\mu,+)}$ and $\widehat{D}_c^{(\mu,-)}$ except in the case $\mu = 1/2$. This result is given in Theorem 3.2, Section 3. The proof is based on Lamperti's time change (2), the Itô-Tanaka formula and the so-called Kennedy martingales. These functionals are also connected to first hitting times, and it is surprising to see in Remarks 2.4 b) and 3.3 how similar the descriptions for $\widehat{D}_0^{(1/2)}$ and $\widehat{D}_c^{(1/2)}$ are. The former can be expressed in terms of first hitting times of BM and the latter in terms of hitting times of BM with drift $1/2$. In a way, the parameter c can be taken into the drift term of $\widehat{D}_c^{(1/2)}$ and letting $c \to 0$ removes the drift leading to the formulas for $\widehat{D}_0^{(1/2)}$.

This paper has been excised from our more complete and more ambitious work [19] under preparation in which we show that many perpetuities, that is, Brownian functionals of the form

$$\int_0^\infty f(B_s^{(\mu)})\, ds$$

may be interpreted as first hitting times of a level by a suitable diffusion; besides, in [19], as well as here, we use Feynman-Kac type arguments to compute, whenever possible, the Laplace transforms of these perpetuities. In [19] we also review many results in the probabilistic literature stating that an integral functional is identical in law with a hitting time; as a famous example, we mention here the Ciesielski-Taylor identities (see Yor [23] p. 50). We refer also to Getoor and Sharpe [7] p. 98, and to Biane [2] (see Remark 4.2) for a generalization to a vast class of pairs of diffusions.

Below, for a given process $\{X_t \,:\, t \geq 0\}$, we often use the notations

$$H_a(X) := \inf\{t \geq 0 \,:\, X_t = a\}$$

and

$$\lambda_a(X) := \sup\{t \geq 0 \,:\, X_t = a\},$$

with the usual conventions $\inf \emptyset = +\infty$ and $\sup \emptyset = 0$.

2 Dufresne's reflected perpetuities

In this section, we compute the joint distribution of the Dufresne's reflected functionals as defined in (5). In fact, we find the joint Laplace transform of the triplet

$$\Big(D^{(\mu,+)}\ ,\ L_\infty^0(B^{(\mu)})\ ,\ D^{(\mu,-)}\Big), \tag{8}$$

where $\mu > 0$ and $L_\infty^0(B^{(\mu)})$ is the total local time of $B^{(\mu)}$ at 0 (normalized to be the occupation time density with respect to the Lebesgue measure). Let, further,

$$D_{-,\lambda_0}^{(\mu,+)} := \int_0^{\lambda_0(B^{(\mu)})} \exp(-2\, B_s^{(\mu)})\, \mathbf{1}_{\{B_s^{(\mu)}>0\}}\, ds,$$

and

$$D^{(\mu,+)}_{\lambda_0,+} := \int_{\lambda_0(B^{(\mu)})}^{\infty} \exp(-2\, B^{(\mu)}_s)\, \mathbf{1}_{\{B^{(\mu)}_s > 0\}}\, ds.$$

Remark 2.1 The term $L^0_\infty(B^{(\mu)})$ is included in our study due to its structural importance arising from the fact that conditionally on $L^0_\infty(B^{(\mu)})$ the variables $D^{(\mu,+)}$ and $D^{(\mu,-)}$ are independent. This property follows, e.g., from the Ray–Knight theorem for Brownian motion with drift and/or excursion theory. We refer also to Pitman and Yor [17] for a study of joint distributions of occupation times and local times for general diffusions.

Theorem 2.2 a) *The joint Laplace transform of Dufresne's reflected perpetuities is, for every $\alpha, \beta > 0$,*

$$\begin{aligned}
&\mathbf{E}_0\Big(\exp\Big(-\alpha\, D^{(\mu,+)} - \beta\, D^{(\mu,-)} \Big)\Big) \\
&\quad = \frac{(\sqrt{2\alpha})^\mu}{\Gamma(\mu+1) 2^\mu I_\mu(\sqrt{2\alpha})} \times \\
&\qquad \times \frac{2\mu\, I_\mu(\sqrt{2\alpha})\, K_\mu(\sqrt{2\beta})}{\sqrt{2\alpha}\, I_{\mu-1}(\sqrt{2\alpha})\, K_\mu(\sqrt{2\beta}) + \sqrt{2\beta}\, I_\mu(\sqrt{2\alpha})\, K_{\mu-1}(\sqrt{2\beta})}.
\end{aligned}$$

b) *The variable $D^{(\mu,+)}_{\lambda_0,+}$ is independent of*

$$\Big(D^{(\mu,+)}_{-,\lambda_0} \,,\; L^0_\infty(B^{(\mu)}) \,,\; D^{(\mu,-)} \Big),$$

and

$$\mathbf{E}_0\Big(\exp(-\alpha\, D^{(\mu,+)}_{\lambda_0,+}) \Big) = \frac{(\sqrt{2\alpha})^\mu}{\Gamma(\mu+1) 2^\mu I_\mu(\sqrt{2\alpha})},$$

$$\begin{aligned}
&\mathbf{E}_0\Big(\exp\Big(-\alpha\, D^{(\mu,+)}_{-,\lambda_0} - \beta\, D^{(\mu,-)} \Big)\Big) \\
&\quad = \frac{2\mu\, I_\mu(\sqrt{2\alpha})\, K_\mu(\sqrt{2\beta})}{\sqrt{2\alpha}\, I_{\mu-1}(\sqrt{2\alpha})\, K_\mu(\sqrt{2\beta}) + \sqrt{2\beta}\, I_\mu(\sqrt{2\alpha})\, K_{\mu-1}(\sqrt{2\beta})}.
\end{aligned}$$

c) *Under $\mathbf{P}_0$, the variable $L^0_\infty(B^{(\mu)})$ is exponentially distributed (with parameter μ). The variables $D^{(\mu,+)}_{-,\lambda_0}$ and $D^{(\mu,-)}$ are conditionally independent* *given* $L^0_\infty(B^{(\mu)})$, *and*

$$\begin{aligned}
&\mathbf{E}\Big(\exp\Big(-\alpha\, D^{(\mu,+)}_{-,\lambda_0} - \beta\, D^{(\mu,-)}\Big)\ \big|\ L^0_\infty(B^{(\mu)}) = u\Big)\\
&\qquad = \exp\Big(-u\,\Big(\Psi_\mu(\sqrt{2\alpha},\sqrt{2\beta}) - \mu\Big)\Big)\\
&\qquad = \exp\Big(-u\,\Big(\Psi_\mu(\sqrt{2\alpha},0) + \Psi_\mu(0,\sqrt{2\beta}) - \mu\Big)\Big),
\end{aligned}$$

where

$$\Psi_\mu(\alpha,0) = \frac{\alpha}{2}\frac{I_{\mu-1}(\alpha)}{I_\mu(\alpha)} \quad \textit{and} \quad \Psi_\mu(0,\beta) = \frac{\beta}{2}\frac{K_{\mu-1}(\beta)}{K_\mu(\beta)}.$$

Proof Let for $\mu > 0$ and $t \geq 0$

$$\widehat{G}^{(\mu)}_t := \exp(-B^{(\mu)}_t), \quad \text{and} \quad \langle \widehat{G}^{(\mu)}\rangle_t := \int_0^t (\widehat{G}^{(\mu)}_s)^2\, ds.$$

With these processes Lamperti's transformation takes the form

$$\widehat{G}^{(\mu)}_t = R^{(\nu)}_{\langle \widehat{G}^{(\mu)}\rangle_t}, \tag{9}$$

where $\nu = -\mu < 0$ and $R^{(\nu)}$ is a Bessel process with index ν started from 1. Applying this transformation we obtain

$$\begin{aligned}
&\Big(D^{(\mu,+)}\ ,\ L^0_\infty(B^{(\mu)})\ ,\ D^{(\mu,-)}\Big) \qquad\qquad (10)\\
&\qquad = \Big(\int_0^{H_0(R^{(\nu)})} \mathbf{1}_{\{R^{(\nu)}_s<1\}}\, ds\ ,\ L^1_{H_0}(R^{(\nu)})\ ,\ \int_0^{H_0(R^{(\nu)})} \mathbf{1}_{\{R^{(\nu)}_s>1\}}\, ds\Big).
\end{aligned}$$

By the well-known time reversal result (see, e.g., formula (2.g) in the paper no. 1 in [24] or II.33 p. 35 in [3]):

$$\{R^{(\nu)}_{H_0-u}\ :\ u \leq H_0(R^{(\nu)})\} \ \stackrel{(\mathrm{d})}{=}\ \{R^{(\mu)}_u\ :\ u \leq \lambda_1(R^{(\mu)})\} \tag{11}$$

where $R^{(\mu)}$ is a BES(μ) process starting from 0. It follows now from (11) and (10) that

$$\begin{aligned}
&\Big(D^{(\mu,+)}\ ,\ L^0_\infty(B^{(\mu)})\ ,\ D^{(\mu,-)}\Big) \qquad\qquad (12)\\
&\qquad \stackrel{(\mathrm{d})}{=} \Big(\int_0^{\lambda_1(R^{(\mu)})} \mathbf{1}_{\{R^{(\mu)}_s<1\}}\, ds\ ,\ L^1_{\lambda_1}(R^{(\mu)})\ ,\ \int_0^{\lambda_1(R^{(\mu)})} \mathbf{1}_{\{R^{(\mu)}_s>1\}}\, ds\Big).
\end{aligned}$$

To conclude the proof of part a), we recall that the joint Laplace-transform of the pair of occupation times

$$\Big(\int_0^{\lambda_1(R^{(\mu)})} \mathbf{1}_{\{R_s^{(\mu)}<1\}}\, ds\ ,\ \int_0^{\lambda_1(R^{(\mu)})} \mathbf{1}_{\{R_s^{(\mu)}>1\}}\, ds.\Big)$$

is computed in Pitman and Yor [16] Proposition 9.2 p. 341. The first claim in part b) follows from the last exit decomposition, and the explicit formulae are, hence, obtained from part a). Finally, part c) is now a plain restatement of the result in Pitman and Yor [16] p. 347–348, see also [17] Section 3. □

Corollary 2.3 *For* $\mu = 1/2$,

$$\mathbf{E}_0\Big(\exp\Big(-\alpha\, D^{(1/2,+)} - \beta\, D^{(1/2,-)}\Big)\Big) = \frac{\sqrt{2\alpha}}{\sqrt{2\alpha}\cosh\sqrt{2\alpha} + \sqrt{2\beta}\sinh\sqrt{2\alpha}}. \tag{13}$$

In particular,

$$\mathbf{E}_0\Big(\exp\Big(-\alpha\, D^{(1/2,+)}\Big)\Big) = \frac{1}{\cosh\sqrt{2\alpha}} \tag{14}$$

and

$$\mathbf{E}_0\Big(\exp\Big(-\beta\, D^{(1/2,-)}\Big)\Big) = \frac{1}{1+\sqrt{2\beta}}. \tag{15}$$

Proof Recalling (see, e.g. Lebedev [11])

$$I_{1/2}(x) = \frac{\sqrt{2}}{\sqrt{\pi x}}\sinh x, \qquad I_{-1/2}(x) = \frac{\sqrt{2}}{\sqrt{\pi x}}\cosh x,$$

and

$$K_{1/2}(x) = K_{-1/2}(x) = \frac{\sqrt{\pi}}{\sqrt{2x}}\,\mathrm{e}^{-x},$$

it is a simple computation to verify the formula (13) from the result in Theorem 2.2 a). □

Remark 2.4 a) Take $\alpha = \beta$ in Theorem 2.2 and use the relationship (see [11])

$$I_{\mu-1}(\sqrt{2\alpha})\, K_\mu(\sqrt{2\alpha}) + I_\mu(\sqrt{2\alpha})\, K_{\mu-1}(\sqrt{2\alpha}) = 1/\sqrt{2\alpha}$$

to obtain

$$\mathbf{E}_0\Big(\exp(-\alpha\,\langle \widehat{G}^{(\mu)}\rangle_\infty)\Big) \;=\; \frac{(2\sqrt{2\alpha})^\mu\, K_\mu(\sqrt{2\alpha})}{\Gamma(\mu)2^\mu} \tag{16}$$

$$\;=\; \mathbf{E}_1\Big(\exp(-\alpha\, H_0(R^{(\mu)}))\Big). \tag{17}$$

From (16) we recover (4) using the classical integral representation of K_μ. The latter equality (17) can be verified by standard diffusion theory (see [3] 4.2.0.1 p. 398). Moreover, for $D^{(\mu,+)}$ we have the description (for a probabilistic explanation, see [19])

$$\int_0^\infty \exp(-2\,B_s^{(\mu)})\,\mathbf{1}_{\{B_s^{(\mu)}>0\}}\,ds \;\stackrel{\text{(d)}}{=}\; H_1(R^{(\mu-1)}),$$

where the Bessel process $R^{(\mu-1)}$ is started at 0 and, in the case $0<\mu<1$, reflected at 0.

b) Letting B be a BM started at 0 and ξ an exponentially (with parameter 1) distributed random variable independent of B we obtain from (15)

$$D^{(1/2,-)} \;\stackrel{\text{(d)}}{=}\; H_\xi(B).$$

Furthermore, from (14) and (13)

$$D^{(1/2,+)} \;\stackrel{\text{(d)}}{=}\; H_1(\tilde{B}),$$

and

$$\langle \widehat{G}^{(1/2)}\rangle_\infty = D^{(1/2,+)} + D^{(1/2,-)} \;\stackrel{\text{(d)}}{=}\; H_1(B),$$

respectively, where B is as above and $\tilde{B}$ is a reflecting BM started from 0 (cf. 3.2.0.1. p. 355 [3]).

3 Dufresne's translated and reflected perpetuities

Recall from (6) the definition of Dufresne's translated perpetuity:

$$\widehat{D}_c^{(\mu)} := \int_0^\infty (c+\exp(B_s^{(\mu)}))^{-2}\,ds.$$

As mentioned in the Introduction we have been able to find an explicit expression for the Laplace transform of $\widehat{D}_c^{(\mu)}$ only when $\mu = 1/2$. This limitation hinges eventually upon the following

Proposition 3.1 *Let $a > 0$ and $\mu > 0$. The process*

$$\Big\{ \log \Big(\frac{a}{a + \exp(B_t^{(-\mu)})} \Big) \ : \ t \geq 0 \Big\}$$

starting from $z = \log\left(a/(a+1)\right)$ has the representation

$$\log \Big(\frac{a}{a + \exp(B_t^{(-\mu)})} \Big) = Z_{I_t}^{(-\mu,\, b)} \tag{18}$$

where

$$I_t = \int_0^t \frac{ds}{(a \exp(-B_s^{(-\mu)}) + 1)^2} \tag{19}$$

and $Z := Z^{(-\mu,\, b)}$ solves the SDE

$$Z_u = z + \beta_u^{(1/2)} - (\frac{1}{2} - \mu) \int_0^u \frac{ds}{1 - \exp(Z_s)},$$

i.e., Z is a diffusion with infinitesimal generator

$$\mathcal{G} = \frac{1}{2} \frac{d^2}{dx^2} + \Big(\frac{1}{2} - \frac{\frac{1}{2} - \mu}{1 - \exp(x)} \Big) \frac{d}{dx}, \quad x < 0. \tag{20}$$

Consequently,

$$\int_0^\infty \frac{ds}{(a \exp(-B_s^{(-\mu)}) + 1)^2} = H_0(Z).$$

Proof By Itô's formula

$$\log\left(a + \exp(B_t^{(-\mu)})\right) - \log(a+1) = \int_0^t \frac{dB_s}{a \exp(-B_s^{(-\mu)}) + 1}$$
$$+ (\frac{1}{2} - \mu) \int_0^t \frac{ds}{a \exp(-B_s^{(-\mu)}) + 1} - \frac{1}{2} \int_0^t \frac{ds}{(a \exp(-B_s^{(-\mu)}) + 1)^2}.$$

Performing here the random time change associated with the additive functional I_t yields the representation (18) with the process Z, as claimed. $\square$

Notice that Z is started at $z < 0$ and killed when it hits 0. We remark that although the drift term tends to $-\infty$, in the case $0 < \mu < 1/2$, as $x \to 0$ the process anyway hits zero. It is, in fact, possible to compute explicitly the scale function and the speed measure of the diffusion Z, and perform the usual boundary point analysis.

Because of the complicated form of the drift term in (20), we have not been able to compute the Laplace transform of $H_0(Z)$, except if $\mu = 1/2$. In this case Z is identical in law to $\{z + \beta_u^{(1/2)} \, : \, u \geq 0\}$ where $\beta^{(1/2)}$ is a BM with drift 1/2. The distributions of the hitting times of any level by $\beta^{(1/2)}$ are, of course, well-known.

To proceed, we take $\mu = 1/2$ and consider the triplet

$$\Big(\widehat{D}_c^{(1/2,+)} \ , \ L_\infty^0(B^{(1/2)}) \ , \ \widehat{D}_c^{(1/2,-)}\Big), \tag{21}$$

where $\widehat{D}_c^{(1/2,\pm)}$ denote Dufresne's translated and reflected functionals as defined in (7). For convenience (cf. the definition of I_t in (19)), we slightly change our notation, and write

$$\Delta_a^{(\pm)} := \int_0^\infty \frac{\mathbf{1}_{\{B_s^{(1/2)} \in \mathbf{R}_\pm\}}}{(a \exp(B_s^{(1/2)}) + 1)^2} \, ds. \tag{22}$$

Clearly,

$$a^2 \, \Delta_a^{(\pm)} = \widehat{D}_{1/a}^{(1/2,\pm)}.$$

Theorem 3.2 *For non-negative α, β and γ*

$$\begin{aligned} F(\alpha,\gamma,\beta) &:= \mathbf{E}_0\Big(\exp\Big(- \alpha \, \Delta_a^{(+)} - \gamma \, L_\infty^0(B^{(1/2)}) - \beta \, \Delta_a^{(-)} \Big)\Big) \\ &= \frac{\sqrt{8\alpha+1} \, \exp(\frac{b}{2})}{\sqrt{8\alpha+1}\cosh(\frac{b}{2}\sqrt{8\alpha+1}) + (2\gamma(a+1) + \sqrt{8\beta+1})\sinh(\frac{b}{2}\sqrt{8\alpha+1})}. \end{aligned} \tag{23}$$

where $b = \log((a+1)/a)(= -z$, with the notation in Proposition 3.1). In particular,

$$\begin{aligned} F(\alpha,0,\alpha) &= \mathbf{E}_0\Big(\exp\Big(- \alpha(\Delta_a^{(+)} + \Delta_a^{(-)})\Big)\Big) \\ &= \exp\Big(- \frac{b}{2}(\sqrt{8\alpha+1} - 1)\Big), \end{aligned} \tag{24}$$

$$F(0,0,\beta) = \mathbf{E}_0\Big(\exp\Big(-\beta\,\Delta_a^{(-)}\Big)\Big) = \frac{1+a}{a+\frac{1}{2}+\sqrt{2\beta+\frac{1}{4}}}, \tag{25}$$

and

$$\begin{aligned} F(\alpha,0,0) &= \mathbf{E}_0\Big(\exp\Big(-\alpha\,\Delta_a^{(+)}\Big)\Big) \\ &= \frac{\sqrt{8\alpha+1}\,\exp(\frac{b}{2})}{\sqrt{8\alpha+1}\cosh(\frac{b}{2}\sqrt{8\alpha+1})+\sinh(\frac{b}{2}\sqrt{8\alpha+1})}. \end{aligned} \tag{26}$$

Moreover, $\Delta_a^{(+)}$ *and* $\Delta_a^{(-)}$ *are conditionally independent given* $L^0_\infty(B^{(1/2)})$ *(cf. Remark 2.1).*

Proof To start with, notice that

$$\Delta_a^{(+)} \overset{\text{(d)}}{=} \int_0^\infty \frac{\exp(2\,B_s^{(-1/2)})\,\mathbf{1}_{\{B_s^{(-1/2)}<0\}}}{(a+\exp(B_s^{(-1/2)}))^2}\,ds.$$

Applying the Lamperti transformation (2) with $\mu=-1/2$ and recalling that $R^{(-1/2)}$ is in fact a BM killed when it hits 0 we obtain

$$\Delta_a^{(+)} \overset{\text{(d)}}{=} \int_0^{H_0(B')} \frac{\mathbf{1}_{\{B'_s\le 1\}}}{(a+B'_s)^2}\,ds,$$

where B' is a BM started at 1. By spatial homogeneity of BM,

$$\Delta_a^{(+)} \overset{\text{(d)}}{=} \int_0^{H_1(B)} \frac{\mathbf{1}_{\{B_s>0\}}}{(a+1-B_s)^2}\,ds,$$

where B is another BM starting from 0. Notice that the transforms we have made for $\Delta_a^{(+)}$ could have been done simultanously for the triplet

$$\Big(\Delta_a^{(+)}\ ,\ L^0_\infty(B^{(1/2)})\ ,\ \Delta_a^{(-)}\Big)$$

which, therefore, is found to be identical in law with

$$\Big(\int_0^{H_1(B)} \frac{\mathbf{1}_{\{B_s>0\}}}{(a+1-B_s)^2}\,ds\ ,\ L^0_{H_1}(B)\ ,\ \int_0^{H_1(B)} \frac{\mathbf{1}_{\{B_s<0\}}}{(a+1-B_s)^2}\,ds\Big). \tag{27}$$

By the Itô-Tanaka formula ($x^+ := \max\{x, 0\}$, $x^- := \max\{-x, 0\}$)

$$\log(a+1-B_t^+) = \log(a+1) - \int_0^t \frac{\mathbf{1}_{\{B_s>0\}}}{(a+1-B_s)}\, dB_s - \frac{L_t^0(B)}{2(a+1)}$$
$$-\frac{1}{2}\int_0^t \frac{\mathbf{1}_{\{B_s>0\}}}{(a+1-B_s)^2}\, ds, \quad \text{for } t < H_{a+1}(B), \tag{28}$$

and

$$\log(a+1+B_t^-) = \log(a+1) - \int_0^t \frac{\mathbf{1}_{\{B_s<0\}}}{(a+1-B_s)}\, dB_s + \frac{L_t^0(B)}{2(a+1)}$$
$$-\frac{1}{2}\int_0^t \frac{\mathbf{1}_{\{B_s<0\}}}{(a+1-B_s)^2}\, ds. \tag{29}$$

In order to apply Skorohod's reflection lemma, we write (28) and (29) in the following equivalent forms

$$-\log(1 - \frac{B_t^+}{a+1}) = -\beta^{(-1/2)}_{I_t^{(+)}} + \frac{L_t^0(B)}{2(a+1)} \tag{30}$$

and

$$\log(1 + \frac{B_t^-}{a+1}) = -\gamma^{(1/2)}_{I_t^{(-)}} + \frac{L_t^0(B)}{2(a+1)}, \tag{31}$$

where $\beta = \{\beta_h^{(-1/2)} \,:\, h \geq 0\}$ and $\gamma = \{\gamma_h^{(1/2)} \,:\, h \geq 0\}$ denote two BM's with drift $-1/2$ and $1/2$, respectively, and

$$I_t^{(\pm)} := \int_0^t \frac{\mathbf{1}_{\{B_s\in\mathbf{R}_\pm\}}}{(a+1-B_s)^2}\, ds.$$

In fact, β and γ are independent by the classical result by Knight on orthogonal continuous martingales (see Revuz and Yor [18] p. 183). Let

$$\sigma_t^{(-1/2)} := \sup_{s\leq t}\{\, \beta_s^{(-1/2)}\,\}.$$

Then Skorohod's lemma applied to (30) gives for $t = H_1(B)$

$$\Big(\int_0^{H_1(B)} \frac{\mathbf{1}_{\{B_s>0\}}}{(a+1-B_s)^2}\, ds \;,\; \frac{L^0_{H_1}(B)}{2(a+1)}\Big) = (\,\theta_b^{(-1/2)}\,,\, \sigma^{(-1/2)}_{\theta_b}\,), \tag{32}$$

where

$$\theta_b^{(-1/2)} = \inf\{u:\ \sigma_u^{(-1/2)} - \beta_u^{(-1/2)} = b\}, \quad b = \log((a+1)/a).$$

Next we consider the third component in (27). From (31) it is seen that

$$\int_0^{H_1(B)} \frac{\mathbf{1}_{\{B_s<0\}}}{(a+1-B_s)^2}\, ds = \inf\Big\{u:\ \gamma_u^{(1/2)} = \frac{L^0_{H_1}(B)}{2(a+1)}\Big\}. \tag{33}$$

But, as we have just seen from (32), $L^0_{H_1}(B)$ is measurable with respect to $\beta^{(-1/2)}$, and, hence, independent of $\gamma^{(1/2)}$. Consequently, because $L^0_{H_1}(B)$ is exponentially distributed with parameter $1/2$, we obtain from (33)

$$\int_0^{H_1(B)} \frac{\mathbf{1}_{\{B_s<0\}}}{(a+1-B_s)^2}\, ds \quad \overset{\text{(d)}}{=} \quad H_\xi(B^{(1/2)}), \tag{34}$$

where $B_0^{(1/2)} = 0$ and ξ is an exponentially (with parameter $(a+1)$) distributed random variable independent of $B^{(1/2)}$, (therefore, (25) holds). Notice also that from (32) and (33) it is immediate that $\Delta_a^{(+)}$ and $\Delta_a^{(-)}$ are conditionally independent given $L^0_\infty(B^{(1/2)})$ (cf. Remark 2.1). In particular,

$$\begin{aligned}\mathbf{E}_0\Big(\exp(-\beta\,\Delta_a^{(-)}) \mid L^0_\infty(B^{(1/2)}) = u\Big) \\ = \exp\Big(-\frac{u}{4(1+a)}(\sqrt{8\beta+1}-1)\Big).\end{aligned} \tag{35}$$

To finish the proof of Theorem 3.2, it now remains essentially to compute the joint Laplace transform of the variables on the right hand side of (32). Firstly, recall that the process

$$K_t^{p,q} = \Big(q\cosh(q(\sigma_t-\beta_t)) + p\sinh(q(\sigma_t-\beta_t))\Big)\,\exp(-p\,\sigma_t - \frac{q^2}{2}\,t),$$

where $\{\beta_t:\ t\geq 0\}$ is a BM starting at 0 and $\sigma_t = \sup\{\beta_s:\ s\leq t\}$, is a martingale for all non-negative p and q starting at q. This is due to Kennedy [10], we refer also to Lehoczky [12] and Azéma and Yor [1], for generalizations and applications. Because $K_t^{p,q}$ is bounded for $t\leq \theta_b := \inf\{t:\ \sigma_t - \beta_t = b\}$ we can use the optional sampling theorem at θ_b to obtain

$$\mathbf{E}\Big(\exp(-p\,\sigma_{\theta_b} - \frac{q^2}{2}\theta_b)\Big) = q\,\Big(q\cosh(q\,b) + p\sinh(q\,b)\Big)^{-1}. \tag{36}$$

By absolute continuity, this yields the corresponding result for a BM with drift μ :

$$
\begin{aligned}
&\mathbf{E}\Big(\exp\big(-p\,\sigma^{(\mu)}_{\theta_b} - \frac{q^2}{2}\theta^{(\mu)}_b\big)\Big) \\
&= \mathbf{E}\Big(\exp\big(\mu\,\beta_{\theta_b} - \frac{\mu^2}{2}\,\theta_b\big)\ \exp\big(-p\,\sigma_{\theta_b} - \frac{q^2}{2}\theta_b\big)\Big) \\
&= \mathbf{E}\Big(\exp\big(-\mu\,(\sigma_{\theta_b} - \beta_{\theta_b}) - (p-\mu)\,\sigma_{\theta_b} - \frac{q^2+\mu^2}{2}\theta_b\big)\Big) \\
&= \exp(-\mu\, b)\mathbf{E}\Big(\exp\big(-(p-\mu)\,\sigma_{\theta_b} - \frac{q^2+\mu^2}{2}\theta_b\big)\Big) \\
&= \frac{\sqrt{q^2+\mu^2}\,\exp(-\mu\, b)}{\sqrt{q^2+\mu^2}\cosh(\sqrt{q^2+\mu^2}\, b) + (p-\mu)\sinh(\sqrt{q^2+\mu^2}\, b)}. \qquad (37)
\end{aligned}
$$

This formula is, in fact, due to Taylor [21]. We refer to Williams [22] for a derivation which is in a manner similar to the one above, see also Pitman and Yor [17]. We are now ready to conclude the computation. By (35), the conditional independence, and (32)

$$
\begin{aligned}
F(\alpha,\gamma,\beta) &= \mathbf{E}_0\Big(\exp\Big(-\alpha\,\Delta^{(+)}_a - \gamma\, L^0_\infty(B^{(1/2)}) - \beta\,\Delta^{(-)}_a\Big)\Big) \\
&= \mathbf{E}_0\Big(\exp\Big(-\alpha\int_0^{H_1(B)} \frac{\mathbf{1}_{\{B_s>0\}}}{(a+1-B_s)^2}\,ds - \hat{\gamma}\,\frac{L^0_{H_1}(B)}{2(1+a)}\Big)\Big) \\
&= \mathbf{E}_0\Big(\exp\Big(-\alpha\,\theta^{(-1/2)}_b - \widehat{\gamma}\,\sigma^{(-1/2)}_{\theta_b}\Big)\Big)
\end{aligned}
$$

where

$$\widehat{\gamma} = 2(1+a)\gamma + \frac{1}{2}(\sqrt{8\beta+1} - 1).$$

Using now (37) with $q = \sqrt{2\alpha}$, $\mu = -1/2$ and $p = \widehat{\gamma}$ gives the claim. □

Remark 3.3 Notice that (24) and (25) are equivalent to

$$\Delta^{(+)}_a + \Delta^{(-)}_a \quad \stackrel{\text{(d)}}{=} \quad H_b(B^{(1/2)}),$$

and

$$\Delta^{(-)}_a \quad \stackrel{\text{(d)}}{=} \quad H_\xi(B^{(1/2)}),$$

respectively, where $B^{(1/2)}$ is started at 0 and ξ is an exponentially with parameter $(1+a)$ distributed random variable independent of B. For a discussion about the functional $\Delta^{(+)}_a$, we refer to [19].

4 Concluding remarks

4.1. Notice that from (13) and (36) it follows that

$$\mathbf{E}_0\Big(\exp\Big(-\alpha\, D^{(1/2,+)}-\beta\, D^{(1/2,-)}\Big)\Big)=\mathbf{E}\Big(\exp(-\alpha\,\theta_1-\sqrt{2\beta}\,\sigma_{\theta_1})\Big).$$

This identity can be explained (and also proved) by proceeding similarly as in the proof of Theorem 3.2 (cf. (28) through (31)). Indeed, we have by the Itô-Tanaka formula

$$\begin{aligned}1-\exp\Big(-(B_t^{(\mu)})^+\Big)=&\int_0^t\exp(-B_s^{(\mu)})\;\mathbf{1}_{\{B_s^{(\mu)}>0\}}\,dB_s^{(\mu)}+\frac{1}{2}L_t^0(B^{(\mu)})\\&-\frac{1}{2}\int_0^t\exp(-B_s^{(\mu)})\;\mathbf{1}_{\{B_s^{(\mu)}>0\}}\,ds\end{aligned}$$

and

$$\begin{aligned}\exp\Big((B_t^{(\mu)})^-\Big)-1=&-\int_0^t\exp(-B_s^{(\mu)})\;\mathbf{1}_{\{B_s^{(\mu)}<0\}}\,dB_s^{(\mu)}+\frac{1}{2}L_t^0(B^{(\mu)})\\&+\frac{1}{2}\int_0^t\exp(-B_s^{(\mu)})\;\mathbf{1}_{\{B_s^{(\mu)}<0\}}\,ds.\end{aligned}$$

Putting here $\mu=1/2$ we deduce (cf. (32)) that

$$\Big(D^{(1/2,+)}\;,\;\frac{1}{2}\,L_\infty^0(B^{(1/2)})\Big)=(\,\theta_1\,,\,\sigma_{\theta_1}\,),$$

and, further, (cf. (33))

$$\begin{aligned}\mathbf{E}_0\Big(\exp(-\beta\, D^{(1/2,-)})\,|\,L_\infty^0(B^{(\mu)})=2\,u\Big)&=\mathbf{E}_0\Big(\exp(-\beta\, H_u(B))\Big)\\&=\exp\Big(-u\sqrt{2\beta}\Big).\end{aligned}$$

4.2. In [2] Biane studies reflected perpetual functionals of a general linear diffusion living on an interval and drifting to the right hand side endpoint of the interval. In particular, for $B^{(\mu)}$ with $\mu>0$ we obtain from [2] Remarque p. 295 that

$$\int_0^\infty g(B_s^{(\mu)})\,\mathbf{1}_{\{B_s^{(\mu)}<0\}}\,ds\;\stackrel{\text{(d)}}{=}\;\int_0^{H_0(X)}g(X_s)\,ds \tag{38}$$

where it is assumed that

(i) g is a positive $\mathcal{C}^1$-function,

(ii) X is a diffusion associated to the generator

$$\frac{1}{2}\frac{d^2}{dx^2} + \left(\mu - \frac{1}{2}\frac{g'(x)}{g(x)}\right)\frac{d}{dx},$$

(iii) $X_t \to +\infty$ a.s. when $t \to \infty$,

(iv) X_0 is exponentially distributed on $(-\infty, 0)$ with parameter 2μ.

Taking in (38) $g(x) = 1$ gives, as discussed in [2], see also Imhof [8], and Doney and Grey [4],

$$\int_0^\infty \mathbf{1}_{\{B_s^{(\mu)}<0\}}\, ds \quad \stackrel{\text{(d)}}{=} \quad H_\xi(B^{(\mu)}),$$

where ξ is exponentially distributed with parameter 2μ independent of $B^{(\mu)}$. Further, letting $g(x) = \exp(2ax)$ the diffusion X in (38) has the generator

$$\frac{1}{2}\frac{d^2}{dx^2} + (\mu - a)\frac{d}{dx},$$

and, hence, the condition (iii) above holds when $\mu > a$. Connections with Biane's and our approaches will be discussed in more details in [19].

4.3. The Laplace transforms of the triplets (8) and (21):

$$\left(D^{(\mu,+)}\ ,\ L^0_\infty(B^{(\mu)})\ ,\ D^{(\mu,-)}\right),$$

and (for $\mu = 1/2$)

$$\left(\widehat{D}_c^{(\mu,+)}\ ,\ L^0_\infty(B^{(\mu)}\ ,\ \widehat{D}_c^{(\mu,-)}\right)$$

could have been computed "directly" using the Feynman-Kac method. In this method the solutions of the associated differential equation are often found by transforming the equation to a "new" equation the solutions of which are known (or consulting Kamke [9]). These transformations are, in a sense, analytic counterparts of random time changes. Indeed, our analysis of the triplets (8) and (21) can, in part, be considered as the probabilistic "viewpoint" of change of variables in Feynman-Kac computations. Moreover, it gives us a good understanding of the structure of the formulas and shows interesting connections between our functionals and some earlier works, as pointed out in the proofs above. In the forthcoming paper [19], we present some general results following this approach, study many examples and discuss also the Feynman-Kac formula for perpetual integral functionals.

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