# Tanaka formula for symmetric Lévy processes

Paavo Salminen[1] and Marc Yor[2]

[1] Åbo Akademi University, Mathematical Department, FIN-20500 Åbo, Finland
phsalmin@abo.fi

[2] Université Pierre et Marie Curie, Laboratoire de Probabilités et Modèles aléatoires, 4, Place Jussieu, Case 188, F-75252 Paris Cedex 05, France

**Summary.** Starting from the potential theoretic definition of the local times of a Markov process – when these exist – we obtain a Tanaka formula for the local times of symmetric Lévy processes. The most interesting case is that of the symmetric $\alpha$-stable Lévy process (for $\alpha \in (1, 2]$) which is studied in detail. In particular, we determine which powers of such a process are semimartingales. These results complete, in a sense, the works by K. Yamada [19] and Fitzsimmons and Getoor [8].

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## 1 Introduction and main results

It is well known that there are different constructions and definitions of local times corresponding to different classes of stochastic processes. For a large panorama of such definitions, see Geman and Horowitz [12].

The most common definition of the local times $L = \{L_t^x : x \in \mathbf{R}, t \geq 0\}$ of a given process $\{X_t : t \geq 0\}$ is as the Radon–Nikodym derivative of the occupation measure of $X$ with respect to the Lebesgue measure in $\mathbf{R}$; precisely $L$ satisfies

$$\int_0^t f(X_s)\, ds = \int_{-\infty}^{\infty} f(x) L_t^x\, dx \tag{1}$$

for every Borel function $f : \mathbf{R} \mapsto \mathbf{R}_+$.

There is also the well known stochastic calculus approach developed by Meyer [16] in which one works with a general semimartingale $\{X_t : t \geq 0\}$, and defines $\Lambda = \{\lambda_t^x : x \in \mathbf{R}, t \geq 0\}$ with respect to the Lebesgue measure from the formula

$$\int_0^t f(X_s)\, d <X^c>_s = \int_{-\infty}^{\infty} f(x) \lambda_t^x\, dx. \tag{2}$$

Of course, in the particular case when $d <X^c>_s = ds$, i.e., $X^c$ is a Brownian motion, then the definitions of $L$ and $\lambda$ coincide. In other cases, e.g., if $X^c \equiv 0$, they will differ.

In this paper we focus on the potential theoretic approach applicable in the Markovian case in which the local times are defined as additive functionals whose $p$-potentials are equal to $p$-resolvent kernels of $X$. Local times can hereby be interpreted as the increasing processes in the Doob-Meyer decompositions of certain submartingales. Considering the $p$-resolvent kernels and passing to the limit, in an adequate manner, as $p \to 0$, we obtain a formula (3), which clearly extends Tanaka's original formula for the local times of Brownian motion to those of the symmetric $\alpha$-stable processes, $\alpha \in (1, 2]$, already obtained by T. Yamada [20] and further developed in K. Yamada [19]. Our approach may be simpler and may help to make these results better known to probabilists working with Lévy processes.

The formula (3) below and its counterparts about decompositions of powers of symmetric $\alpha$-stable Lévy processes show at the same time similarities and differences with the well known formulae for Brownian motion (see, in particular, Chapter 10 in [23] concerning the principal values of Brownian local times). We hope that the Tanaka representation of the local times in (3) may be useful to gain some better understanding for the Ray-Knight theorems of the local times of $X$ as presented in Eisenbaum et al. [6], since in the Brownian case, Tanaka's formula has been such a powerful tool for this purpose, see, e.g., Jeulin [15].

We now state the main formulae and results for the symmetric $\alpha$-stable Lévy process $X = \{X_t\}$. To be precise, we take $X$ to satisfy

$$\mathbf{E}\left(\exp(\mathrm{i}\lambda X_t)\right) = \exp(-t|\lambda|^\alpha), \quad \lambda \in \mathbf{R},$$

in particular, for $\alpha = 2$, $X$ equals $\sqrt{2}$ times a standard BM. General criteria can be applied to verify that $X$ possesses a jointly continuous family of local times $\{L_t^x\}$ satisfying (1). The constants $c_i$ appearing below and later in the paper will be computed precisely in Section 5; clearly, they depend on the index $\alpha$ and/or the exponent $\gamma$.

**1)** *For all $t \geq 0$ and $x \in \mathbf{R}$*

$$|X_t - x|^{\alpha-1} = |x|^{\alpha-1} + N_t^x + c_1\, L_t^x, \tag{3}$$

*where $N^x$ is a martingale such that for $0 \leq \gamma < \alpha/(\alpha-1)$, especially for $\gamma = 2$,*

$$\mathbf{E}\left(\sup_{s\leq t} |N_s^x|^\gamma\right) < \infty. \tag{4}$$

*Moreover, the continuous increasing process associated with $N^x$ is*

$$<N^x>_t := c_2 \int_0^t \frac{ds}{|X_s - x|^{2-\alpha}}. \tag{5}$$

**2)** *For $\alpha - 1 < \gamma < \alpha$ the submartingale $\{|X_t - x|^\gamma\}$ has the decomposition*

$$|X_t - x|^\gamma = |x|^\gamma + N_t^{(\gamma)} + A_t^{(\gamma)}, \tag{6}$$

*where $N^{(\gamma)}$ is a martingale and $A^{(\gamma)}$ is the increasing process given by*

$$A_t^{(\gamma)} := c_3 \int_0^t \frac{ds}{|X_s - x|^{\alpha-\gamma}}. \tag{7}$$

**3)** *For $0 < \gamma < \alpha - 1$ the process $\{|X_t - x|^\gamma\}$ is not a semimartingale but for $(\alpha-1)/2 < \gamma < \alpha-1$ it is a Dirichlet process with the canonical decomposition*

$$|X_t - x|^\gamma = |x|^\gamma + N_t^{(\gamma)} + A_t^{(\gamma)}, \tag{8}$$

*where $N^{(\gamma)}$ is a martingale and $A^{(\gamma)}$, which has zero quadratic variation, is given by the principal value integral*

$$A_t^{(\gamma)} := c_4 \text{ p.v.} \int_0^t \frac{ds}{|X_s - x|^{\alpha-\gamma}} := c_4 \int \frac{dz}{|z|^{\alpha-\gamma}} \left( L_t^{x+z} - L_t^{x-z} \right). \tag{9}$$

The paper is organized so that in Section 2 some preliminaries about symmetric Lévy processes including their generators and some variants of the Itô formula are presented. In Section 3 we derive the Tanaka formula for general symmetric Lévy processes admitting local times. The above stated results for symmetric stable Lévy processes are proved and extended in Section 4. In Section 5 we compute explicitly the constants $c_i$ featured above and also further ones appearing especially in Section 4. This is done by exhibiting some close relations between these constants and the known expressions of the moments $\mathbf{E}(|X_1|^\gamma)$ where $X_1$ denotes a standard symmetric $\alpha$-stable variable. In Section 6, we consider, instead of $|X_t - x|^\gamma$, the process $\{(X_t - x)^{\gamma,*}\}$, where

$$a^{\gamma,*} := \operatorname{sgn}(a)\, |a|^\gamma,$$

is the symmetric power of order $\gamma$, and we determine the parameter values for which these processes are semimartingales or Dirichlet processes, thus completing results 1), 2) and 3) above.

## 2 Preliminaries on symmetric Lévy processes

Throughout this paper, we consider a real-valued symmetric Lévy process $X = \{X_t\}$ and, if nothing else is stated, we assume $X_0 = 0$. The Lévy exponent $\Psi$ of $X$ is a non-negative symmetric function such that

$$\mathbf{E}\left(\exp(\mathrm{i}\xi X_t)\right) = \mathbf{E}\left(\cos(\xi X_t)\right) = \exp\left(-t\Psi(\xi)\right). \tag{10}$$

The Lévy measure $\nu$ of $X$ satisfies, as is well known, the integrability condition

$$\int_{-\infty}^{\infty} (1 \wedge z^2)\, \nu(dz) < \infty.$$

By symmetry, $\nu(A) = \nu(-A)$ for any $A \in \mathcal{B}$, the Borel $\sigma$-field on $\mathbf{R}$; hence,

$$\begin{aligned}\Psi(\xi) &= \frac{1}{2}\,\sigma^2\,\xi^2 - \int_{-\infty}^{\infty} \left(\mathrm{e}^{\mathrm{i}\,\xi\,z} - 1 - \mathrm{i}\,\xi\,z\,\mathbf{1}_{\{|z|\le 1\}}\right)\,\nu(dz) \\ &= \frac{1}{2}\,\sigma^2\,\xi^2 + 2\int_0^{\infty} (1 - \cos(\xi z))\,\nu(dz). \end{aligned} \tag{11}$$

Recall also (see, e.g., Ikeda and Watanabe [14] p. 65) that $X$ admits the Brownian-Poisson representation

$$X_t = \sigma\,B_t + \int_{(0,t]}\int_{\{|z|\ge 1\}} z\,\Pi(ds,dz) + \int_{(0,t]}\int_{\{|z|<1\}} z\,(\Pi - \pi)(ds,dz), \tag{12}$$

where the Brownian motion $B$ and the Poisson random measure $\Pi$ with the intensity

$$\pi(ds,dz) := \mathbf{E}(\Pi(ds,dz)) = ds\,\nu(dz)$$

are independent. Due to the symmetry of $\nu$, the generator of $X$ can be written as

$$\begin{aligned}\mathcal{G}f(x) &:= \mathcal{G}^B f(x) + \mathcal{G}^{\Pi} f(x) \\ &:= \frac{1}{2}\,\sigma^2\,f''(x) + \int_{\mathbf{R}} \left(f(x+y) - f(x) - f'(x)\,y\,\mathbf{1}_{\{|y|<1\}}\right)\,\nu(dy) \\ &= \frac{1}{2}\,\sigma^2\,f''(x) + \int_{\mathbf{R}} \left(f(x+y) - f(x) - f'(x)\,y\right)\,\nu(dy). \end{aligned} \tag{13}$$

where $\mathcal{G}$ acts on regular functions $f$ in particular those in the Schwartz space $S(\mathbf{R})$ of rapidly decreasing functions. Given a smooth function $f$, the predictable form of the Itô formula (see Ikeda and Watanabe [14] and K. Yamada [19]) writes

$$\begin{aligned} f(X_t) - f(X_0) - \int_0^t \mathcal{G}f(X_s)\,ds \qquad\qquad & \qquad (14)\\ = \sigma\int_0^t f'(X_s)\,dB_s + \int_0^t \int_{\mathbf{R}} \left(f(X_{s-}+z) - f(X_{s-})\right)\,(\Pi - \pi)(ds,dz). & \end{aligned}$$

The formula (14) connects with the Itô formula for semimartingales, as developed by Meyer [16], and displayed as

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})\,dX_s + \frac{\sigma^2}{2}\int_0^t f''(X_s)\,ds \\ + \sum_{0<s\le t} \left(f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\right). \tag{15}$$

The sum of jumps $\sum_{0<s\le t}(\dots)$ may be compensated by $\int_0^t \mathcal{G}^{\Pi} f(X_s)\,ds$, and, hence, we have recovered the integrated form of (13):

$$\int_0^t \mathcal{G} f(X_s)\,ds = \int_0^t \mathcal{G}^B f(X_s)\,ds + \int_0^t \mathcal{G}^{\Pi} f(X_s)\,ds.$$

We record also a more general compensator formula employed later in the paper. For this, let $\Phi : \mathbf{R}\times\mathbf{R} \mapsto \mathbf{R}_+$ be a Borel measurable function. Then

$$\mathbf{E}\left(\sum_{0<s\le t} \Phi(X_{s-}, X_s)\mathbf{1}_{\{\Delta X_s \neq 0\}}\right) \\ = \mathbf{E}\left(\int_0^t \int_{\mathbf{R}\setminus\{0\}} \pi(ds, dz)\Phi(X_s, X_s + z)\right). \tag{16}$$

## 3 Local times for symmetric Lévy processes

From now on, we assume that

$$\int_{-\infty}^{\infty} \frac{1}{1+\Psi(\xi)}\,d\xi < \infty. \tag{17}$$

From standard Fourier arguments (see Bertoin [1] and, e.g., Borodin and Ibragimov [2] p. 67) one can show the existence of a jointly measurable family of local times $\{L_t^x : x \in \mathbf{R}, t \ge 0\}$ satisfying for every Borel-measurable function $f : \mathbf{R} \mapsto \mathbf{R}_+$ the occupation time formula

$$\int_0^t ds\, f(X_s) = \int_{-\infty}^{\infty} f(x) L_t^x\,dx.$$

For the condition (expressed in terms of the function $v$ in (22)) under which $(t,x) \mapsto L_t^x$ is continuous, see Bertoin [1] p. 148. In particular, the condition holds for symmetric $\alpha$-stable Lévy processes; in fact it was shown by Boylan [3], see also Getoor and Kesten [13], that

$$|L_t^{x+y} - L_t^x| \le K_t\,|y|^{\theta} \tag{18}$$

for any $\theta < (\alpha-1)/2$ and some random constant $K_t$.

Our approach toward a Tanaka formula for these local times is based on the potential theoretic construction which we now develop. It is well known, see Bertoin [1] p. 67, that for any $p > 0$

$$u^{(p)}(x) = \frac{1}{\pi} \int_0^\infty \frac{\cos(\xi x)}{p + \Psi(\xi)}\, d\xi \tag{19}$$

is a continuous version of the density of the resolvent

$$U^{(p)}(0, dx) = \mathbf{E}_0 \left( \int_0^\infty \mathrm{e}^{-p\,t} \mathbf{1}_{\{X_t \in dx\}}\, dt \right).$$

Moreover, for every $x$ the local time $\{L_t^x\}$ can be chosen as a continuous additive functional such that

$$u^{(p)}(y - x) = \mathbf{E}_y \left( \int_0^\infty \mathrm{e}^{-p\,t} d_t L_t^x \right). \tag{20}$$

From (20) we deduce the Doob-Meyer decomposition given in the next

**Proposition 1.** *For every fixed $x$*

$$u^{(p)}(X_t - x) = u^{(p)}(X_0 - x) + M_t^{(p,x)} + p \int_0^t u^{(p)}(X_s - x) ds - L_t^x, \tag{21}$$

*where $M^{(p,x)}$ is a martingale with respect to the natural filtration $\{\mathcal{F}_t\}$ of $X$. Moreover, for every fixed $t$, both the martingale $\{M_s^{(p,x)} : s \le t\}$ and the random variable $L_t^x$ belong to BMO; in particular, $L_t^x$ has some exponential moments.*

*Proof.* Straightforward computations using the Markov property show that for $y = X_0$

$$\mathbf{E}_y \left( \int_0^\infty \mathrm{e}^{-p\,t} d_t L_t^x \,|\, \mathcal{F}_s \right) = \int_0^s \mathrm{e}^{-p\,t} d_t L_t^x + \mathrm{e}^{-p\,s} u^{(p)}(X_s - x),$$

which together with an integration by parts yields (21). We leave the proofs of the remaining assertions to the reader.

A variant of the Tanaka formula shall now be obtained by letting $p \to 0$ in (21). The result is stated in Proposition 2 but first we need an important ingredient.

**Lemma 1.** *For every $x \in \mathbf{R}$*

$$\lim_{p \to 0} \left( u^{(p)}(0) - u^{(p)}(x) \right) = \frac{1}{\pi} \int_0^\infty \frac{1 - \cos(\xi x)}{\Psi(\xi)}\, d\xi =: v(x). \tag{22}$$

*Proof.* The statement follows from (19) by dominated convergence because (cf. (11))

$$\int_1^\infty \frac{1}{\Psi(\xi)}\,d\xi < \infty, \quad \text{and} \quad \int_0^1 \frac{\xi^2}{\Psi(\xi)}\,d\xi < \infty.$$

Notice also that $v$ is continuous.

The formula (23) below generalizes in a sense the Tanaka formula for Brownian motion to symmetric Lévy processes. In the next section we study the particular case of symmetric stable processes.

**Proposition 2.** *Let $v$ be the function introduced in (22) and $M^{(p,x)}$ the martingale defined in Proposition 1. Then*

$$v(X_t - x) = v(x) + \tilde{N}_t^x + L_t^x, \tag{23}$$

*where $\tilde{N}_t^x := -\lim_{p\to 0} M_t^{(p,x)}$ defines a martingale.*

*Remark 1.* Standard results about martingale additive functionals of $X$ yield the following representations

$$\begin{aligned}\tilde{N}_t^x = \sigma \int_0^t v'(X_s - x)\,dB_s \\ + \int_{(0,t]}\int_{\mathbf{R}} \left(v(X_{s-} - x + z) - v(X_{s-} - x)\right)\ (\Pi - \pi)(ds, dz),\end{aligned}$$

and

$$\begin{aligned}< \tilde{N}^x >_t = \sigma^2 \int_0^t (v'(X_s - x))^2\,ds \\ + \int_0^t \int_{\mathbf{R}} \left(v(X_s - x + z) - v(X_s - x)\right)^2\ \pi(ds,\, dz),\end{aligned}$$

where $v'$ is a weak derivative of $v$.

*Proof.* Consider the identity (21). Let therein $p \to 0$ and use Lemma 1 to obtain

$$v(X_t - x) = v(x) - \lim_{p\to 0}\left(M_t^{(p,x)} + p\int_0^t u^{(p)}(X_s - x)ds\right) + L_t^x. \tag{24}$$

From (19) $u^{(p)}(y) \le u^{(p)}(0)$, and, consequently,

$$0 \le p\int_0^t u^{(p)}(X_s - x)ds \le p\,u^{(p)}(0)\,t. \tag{25}$$

Next we show that

$$\lim_{p\to 0} p\,u^{(p)}(0) = 0. \tag{26}$$

Indeed, using (19) again,

$$\begin{aligned} p\,u^{(p)}(0) &= \frac{1}{\pi}\int_0^\infty \frac{p\,d\xi}{p+\Psi(\xi)} \\ &\le \frac{1}{\pi}\int_0^1 \frac{p\,d\xi}{p+\Psi(\xi)} + \frac{p}{\pi}\int_1^\infty \frac{d\xi}{\Psi(\xi)}, \end{aligned} \tag{27}$$

and (26) results by dominated convergence. Hence, (24) yields (23) with $\tilde{N}^x$ as claimed. It remains to prove that $\tilde{N}^x$ is a martingale. For this it is enough to show that

$$\mathbf{E}\left(|\tilde{N}_t^x - M_t^{(p,x)}|\right) \to 0 \quad \text{as } p \to 0. \tag{28}$$

To prove (28) consider

$$\begin{aligned} |\tilde{N}_t^x - M_t^{(p,x)}| \le p\int_0^t u^{(p)}(X_s - x)ds + |v(x) - (u^{(p)}(0) - u^{(p)}(x))| \\ +|v(X_t - x) - (u^{(p)}(0) - u^{(p)}(X_t - x))|. \end{aligned}$$

From (25) and (26), the integral term goes to 0 as $p \to 0$. Next, by Fubini's theorem and (10)

$$\begin{aligned} &\mathbf{E}\left(\left|v(X_t - x) - (u^{(p)}(0) - u^{(p)}(X_t - x))\right|\right) \\ &\quad = \frac{1}{\pi}\,p\int_0^\infty \frac{1 - \mathbf{E}(\cos(\xi(X_t - x)))}{\Psi(\xi)(p+\Psi(\xi))}\,d\xi \\ &\quad = \frac{1}{\pi}\,p\int_0^\infty \frac{1 - \cos(\xi\,x)\,\exp(-t\Psi(\xi))}{\Psi(\xi)(p+\Psi(\xi))}\,d\xi \\ &\quad \le \frac{1}{\pi}\,p\left(\int_0^\infty \frac{1 - \cos(\xi\,x)}{\Psi(\xi)(p+\Psi(\xi))}\,d\xi + \int_0^\infty \frac{t\,\Psi(\xi)}{\Psi(\xi)(p+\Psi(\xi))}\,d\xi\right). \end{aligned}$$

Applying the dominated convergence theorem for the first term above and (27) for the second one give

$$\lim_{p\to 0}\mathbf{E}\left(\left|v(X_t - x) - (u^{(p)}(0) - u^{(p)}(X_t - x))\right|\right) = 0,$$

completing the proof.

*Example 1.* For standard Brownian motion $B$ we have

$$u^{(p)}(x) = \frac{1}{\sqrt{2p}}\,\mathrm{e}^{-\sqrt{2p}|x|}.$$

Consequently,

$$v(x) := \lim_{p\to 0}\left(u^{(p)}(0) - u^{(p)}(x)\right) = |x|.$$

and the formula (23) takes the familiar form

$$|B_t - x| = |x| + N_t^x + L_t^x$$

where

$$N_t^x = \lim_{p\to 0} \int_0^t \mathrm{e}^{-\sqrt{2p}|B_s - x|} \operatorname{sgn}(B_s - x)\, dB_s$$
$$= \int_0^t \operatorname{sgn}(B_s - x)\, dB_s.$$

## 4 Symmetric $\alpha$-stable Lévy processes

Let $X = \{X_t\}$, $X_0 = 0$, denote the symmetric $\alpha$-stable process with the Lévy exponent

$$\Psi(\xi) = |\xi|^\alpha, \quad \alpha \in (1,2).$$

We remark that the condition (17) is satisfied, and also that the local time of $X$ has a jointly continuous version, as is discussed in Section 3. For clarity, we have excluded the Brownian motion from our study. However, the corresponding results for Brownian motion may be recovered by letting $\alpha \to 2$. Recall also that $\mathbf{E}(|X_t|^\gamma) < \infty$ for $\gamma < \alpha$, and that the Lévy measure is

$$\nu(dz) = c_5(\alpha)\, |z|^{-\alpha-1}\, dz, \quad \alpha \in (1,2). \tag{29}$$

The function $v$ introduced in Lemma 1 is in the present case given by

$$v(x) = c_6(\alpha)\, |x|^{\alpha-1}. \tag{30}$$

The results announced in the Introduction are now presented again and proven in a more complete form through the following three propositions. The first one treats the claim 1) in the Introduction.

**Proposition 3. a)** *For fixed* $x$

$$c_6(\alpha)\, \left(|X_t - x|^{\alpha-1} - |x|^{\alpha-1}\right) = \tilde{N}_t^x + L_t^x \tag{31}$$

*where* $\{\tilde{N}_t^x\}$ *is a square integrable martingale. In fact, for all* $0 \le \gamma < \alpha/(\alpha - 1)$, *especially for* $\gamma = 2$,

$$\mathbf{E}\left(\sup_{s\le t} |\tilde{N}_s^x|^\gamma\right) < \infty. \tag{32}$$

*Moreover, the continuous increasing process associated with* $\tilde{N}^x$ *is*

$$<\tilde{N}^x>_t := c_7(\alpha) \int_0^t \frac{ds}{|X_s - x|^{2-\alpha}}. \tag{33}$$

**b)** *For every* $t$ *and* $x$ *the variable* $L_t^x$ *belongs to BMO; in fact, for all* $s \le t$

$$\mathbf{E}\,(L_t^x - L_s^x \,|\, \mathcal{F}_s) \le K_{\alpha,t} \tag{34}$$

*for some constant* $K_{\alpha,t}$ *which does not depend on* $s$.

*Proof.* The fact that $\tilde{N}^x$ is a martingale is clear from Proposition 2. Because $L^x_t$ has some exponential moments (cf. Proposition 1), it is seen easily from(31) that for $\gamma > 0$

$$\mathbf{E}(|\tilde{N}^x_t|^\gamma) < \infty$$

if

$$\mathbf{E}\left(|X_t - x|^{\gamma(\alpha-1)}\right) < \infty,$$

which is true for $\gamma(\alpha - 1) < \alpha$. Consequently, an extension of the Doob-Kolmogorov inequality, gives (32). The martingale $\tilde{N}^x$ has no continuous martingale part. Hence, letting

$$[\tilde{N}^x]_t := \sum_{s\le t}(\varDelta\tilde{N}^x_s)^2 := (c_6(\alpha))^2 \sum_{s\le t}\left(|X_s - x|^{\alpha-1} - |X_{s-} - x|^{\alpha-1}\right)^2$$

it holds that $\{\,(\tilde{N}^x_t)^2 - [\tilde{N}^x]_t\,\}$ is a martingale. Consequently, $<\tilde{N}^x>$ can be obtained as the dual predictable projection of $[\tilde{N}^x]$, and from the Lévy system of $X$, e.g., (16), we get

$$<\tilde{N}^x>_t = (c_6(\alpha))^2\, c_5(\alpha) \int_0^t ds \int_{\mathbf{R}} \frac{dy}{|y|^{\alpha+1}} \left(|X_{s-} - x + y|^{\alpha-1} - |X_{s-} - x|^{\alpha-1}\right)^2 .$$

Putting $z = X_{s-} - x$ and introducing $y = zu$ the latter integral takes the form

$$\int_{\mathbf{R}} \frac{dy}{|y|^{\alpha+1}} \left(|z + y|^{\alpha-1} - |z|^{\alpha-1}\right)^2 = \frac{1}{|z|^{2-\alpha}} \int_{\mathbf{R}} \frac{du}{|u|^{\alpha+1}} \left(|1 + u|^{\alpha-1} - 1\right)^2 .$$

Consequently, $<\tilde{N}^x>$ is as claimed. To prove the second part of the proposition, notice that by the martingale property

$$\begin{aligned}
\mathbf{E}\left(L^x_t - L^x_s \,|\, \mathcal{F}_s\right) &= c_6(\alpha)\ \mathbf{E}\left(|X_t - x|^{\alpha-1} - |X_s - x|^{\alpha-1} \,|\, \mathcal{F}_s\right)\\
&\le c_6(\alpha)\ \mathbf{E}\left(|X_t - X_s|^{\alpha-1} \,|\, \mathcal{F}_s\right)\\
&\le c_6(\alpha)\ \mathbf{E}\left(|X_{t-s}|^{\alpha-1}\right)\\
&\le K'_\alpha\, t^{(\alpha-1)/\alpha},
\end{aligned}$$

where also the scaling property and the inequality

$$|x^p - y^p| \le |x - y|^p, \quad 0 < p \le 1,$$

are used.

The following corollary plays the same rôle for $X$ as the classical Itô-Tanaka formula plays for Brownian motion. In fact, a large part of this paper discusses for which functions the identity (35), or some variant of it is valid.

**Corollary 1.** *Let $f$ be a bounded Borel function with compact support and define*

$$F(y) := \int dx f(x)\, |y-x|^{\alpha-1}.$$

*Then*

$$F(X_t) = F(0) + \int dx f(x)\, N_t^x + c_1(\alpha) \int_0^t ds\, f(X_s) \tag{35}$$

*expresses the canonical semimartingale decomposition of $\{F(X_t)\}$ with $\{\int dx f(x)\, N_t^x\}$ a martingale.*

*Proof.* It suffices to integrate both sides of (31) (or rather (3)) with respect to the measure $f(x)\, dx$.

*Remark 2.* **a)** In K. Yamada [19] the representation (31) of the local time (or Tanaka's formula for symmetric $\alpha$-stable processes) is derived using the so called "mollifier" approach as in Ikeda and Watanabe [14] in the Brownian motion case. In this case the martingale is given by

$$\tilde{N}_t^x = c_6(\alpha) \int_{(0,t]} \int_{\mathbf{R}} \left( |X_{s-} - x + z|^{\alpha-1} - |X_{s-} - x|^{\alpha-1} \right) \, (\Pi - \pi)(ds, dz),$$

where $\Pi$ and $\pi$ are the Poisson random measure and the corresponding intensity measure, respectively, associated with $X$.
**b)** The inequality (34) holds for all symmetric Lévy processes having local times. Indeed, it is proved in Bertoin [1] p. 147 Corollary 14 that the function $v$ defined in (22), Lemma 1, induces a metric on $\mathbf{R}$, and, in particular, the triangle inequality holds. Consequently,

$$\mathbf{E}\left(L_t^x - L_s^x \,|\, \mathcal{F}_s\right) \le \mathbf{E}(v(X_t - X_s)) = \mathbf{E}(v(X_{t-s})) \le \mathbf{E}(v(X_t)) < \infty$$

because

$$\begin{aligned} \mathbf{E}(v(X_t)) &= \frac{1}{\pi} \int_0^\infty \frac{1 - \exp(-t\Psi(\xi))}{\Psi(\xi)}\, d\xi \\ &\le \frac{1}{\pi} \left( t + \int_1^\infty \frac{d\xi}{\Psi(\xi)} \right) < \infty. \end{aligned}$$

**c)** We leave it to the reader to establish a version of Corollary 1 for general symmetric Lévy processes.

**Proposition 4.** *For a given $x$ and $\alpha - 1 < \gamma < \alpha$ the submartingale $\{|X_t - x|^\gamma : t \ge 0\}$ has the decomposition*

$$|X_t - x|^\gamma = |x|^\gamma + N_t^{(\gamma)} + A_t^{(\gamma)}, \tag{36}$$

*where $N^{(\gamma)}$ is a martingale and $A^{(\gamma)}$ is the increasing process given by*

$$A_t^{(\gamma)} = c_3(\alpha,\gamma) \int_0^t \frac{ds}{|X_s - x|^{\alpha-\gamma}}. \tag{37}$$

*Moreover, when* $\alpha - 1 \le \gamma \le \alpha/2$ *the increasing process* $<N^{(\gamma)}>$ *is of the form*

$$<N^{(\gamma)}>_t = c_8(\alpha,\gamma) \int_0^t \frac{ds}{|X_s - x|^{\alpha-2\gamma}}. \tag{38}$$

*Proof.* Formula (36) is obtained by integrating both sides of equation (31) (or (3) taken at level $z$ with respect to the measure $dz/|z-x|^{\alpha-\gamma}$. The form of the left hand side is obtained from the scaling argument. Because $A^{(\gamma)}$ is continuous the computation for finding $< N^{(\gamma)} >$ is very similar to the computation of $<\tilde{N}^x>$ in the proof of Proposition 3. We have

$$<N^{(\gamma)}>_t = \int_0^t ds \int_{\mathbf{R}} \nu(dy) \left(|X_{s-} - x + y|^\gamma - |X_{s-} - x|^\gamma\right)^2, \tag{39}$$

which easily yields (38).

For the next proposition, we recall the notion of Dirichlet process, that is a process which can decomposed uniquely as the sum of a local martingale and a continuous process with zero quadratic variation (see, e.g., Föllmer [10], Fukushima [11]).

**Proposition 5. a)** *For* $0 < \gamma < \alpha - 1$ *the process* $|X - x|^\gamma$ *is not a semimartingale.*
**b)** *For* $(\alpha-1)/2 < \gamma < \alpha - 1$ *the process* $|X - x|^\gamma$ *is a Dirichlet process with the canonical decomposition*

$$|X_t - x|^\gamma = |x|^\gamma + N_t^{(\gamma)} + A_t^{(\gamma)}, \tag{40}$$

*where* $N^{(\gamma)}$ *is a martingale and* $A^{(\gamma)}$ *is given by the principal value integral*

$$\begin{aligned} A_t^{(\gamma)} &= c_4(\alpha,\gamma) \text{ p.v.} \int_0^t \frac{ds}{|X_s - x|^{\alpha-\gamma}} \\ &= c_4(\alpha,\gamma) \int_{\mathbf{R}} \frac{dz}{|z|^{\alpha-\gamma}} \left(L_t^{x+z} - L_t^{x-z}\right). \end{aligned} \tag{41}$$

*Moreover, the increasing process* $<N^{(\gamma)}>$ *is as given in (38).*

*Proof.* a) We take $x = 0$ and adapt the argument in Yor [21] applied therein for continuous martingales. Assume that $Y_t := |X_t|^\gamma$, $\gamma < \alpha - 1$, defines a semimartingale. Then

$$|X_t|^{\alpha-1} = Y_t^{\,\theta}$$

with $\theta = \gamma/(\alpha-1) > 1$, and Itô's formula for semimartingales (notice that $Y^c \equiv 0$) gives

$$Y_t^\theta = \int_0^t \theta\, Y_{s-}^{\theta-1}\, dY_s + \Sigma_t, \tag{42}$$

where

$$\Sigma_t := \sum_{0<s\le t} \left( Y_s^\theta - Y_{s-}^\theta - \theta\, Y_{s-}^{\theta-1} \Delta Y_s \right).$$

The argument of the proof is that under the above assumption the local time

$$L_t^0 \equiv \int_0^t \mathbf{1}_{\{X_{s-}=0\}}\, d|X_s|^{\alpha-1} \tag{43}$$

would be equal to zero. To derive this contradiction notice from (42) and (43) that

$$L_t^0 = \int_0^t \mathbf{1}_{\{Y_{s-}=0\}}\, dY_s^\theta = \int_0^t \mathbf{1}_{\{Y_{s-}=0\}}\, d\Sigma_s.$$

But because $\Sigma$ is a purely discontinuous increasing process and $L^0$ is continuous this is possible only if $L^0 \equiv 0$, which cannot be the case; thus proving that $Y$ is not a semimartingale.

b) To prove (40) we consider formula (3) at levels $x+z$ and $x-z$ and write

$$\begin{aligned} &\int_{\mathbf{R}} \frac{dz}{|z|^{\alpha-\gamma}} \left( |X_t-(x+z)|^{\alpha-1} - |X_t-(x-z)|^{\alpha-1} \right) \qquad (44)\\ &\qquad = \int_{\mathbf{R}} \frac{dz}{|z|^{\alpha-\gamma}} \left( N_t^{x+z} - N_t^{x-z} \right) + c_1(\alpha) \int_{\mathbf{R}} \frac{dz}{|z|^{\alpha-\gamma}} \left( L_t^{x+z} - L_t^{x-z} \right). \end{aligned}$$

The integral on the left hand side is well defined since by scaling

$$\int_{\mathbf{R}} \frac{dz}{|z|^{\alpha-\gamma}} \left( |X_t-(x+z)|^{\alpha-1} - |X_t-(x-z)|^{\alpha-1} \right) = |X_t - x|^\gamma\, r(\alpha,\gamma)$$

with

$$r(\alpha,\gamma) := \int_{\mathbf{R}} \frac{dz}{|z|^{\alpha-\gamma}} \left( |1-z|^{\alpha-1} - |1+z|^{\alpha-1} \right),$$

which is an absolutely convergent integral. Next notice that the principal value integral on the right hand side of (44) is well defined by the Hölder continuity in $x$ of the local times (cf. (18)). It also follows that the first integral on the right hand side of (44) is meaningful and, by Fubini's theorem, it is a martingale. In Fitzsimmons and Getoor [8] it is proved that

$$H_t^0 := \int_0^\infty \frac{dz}{z^{\alpha-\gamma}} \left( L_t^{-z} - L_t^0 \right).$$

has zero $p$-variation for $p > p_o := (\alpha-1)/\gamma$ (notice $1+\gamma$ in [8] corresponds ours $\alpha-\gamma$). Since $p_o < 2$ it is now easily seen that also $\{A_t^{(\gamma)}\}$ has zero quadratic variation and the claimed Dirichlet process decomposition follows with

$$c_4(\alpha,\gamma) = c_1(\alpha)/r(\alpha,\gamma). \tag{45}$$

## 5 Explicit values of the constants

An important ingredient in the computation of the explicit values of the constants is the formula for absolute moments of symmetric $\alpha$-stable, $\alpha \in (1,2)$, random variables due to Shanbhag and Sreehari [18] (see also Sato [17] p. 163, Chaumont and Yor [4] p. 110). To discuss this briefly let

- $Z$ be an exponentially distributed r.v. with mean 1,
- $U$ a normally distributed r.v. with mean 0 and variance 1,
- $X^{(\alpha)}$ a symmetric $\alpha$-stable r.v. with characteristic function $\exp(-|\xi|^{\alpha})$,
- $Y^{(\alpha/2)}$ a positive $\alpha/2$-stable r.v. with Laplace transform $\exp(-\xi^{\alpha/2})$.

Assume also that these variables are independent. Then it is easily checked that

$$\left(Z/Y^{(\alpha/2)}\right)^{\alpha/2} \stackrel{d}{=} Z \tag{46}$$

and

$$X^{(\alpha)} \stackrel{d}{=} \sqrt{2}\,U\,\left(Y^{(\alpha/2)}\right)^{1/2}. \tag{47}$$

From (46) we obtain for $\gamma < \alpha/2$

$$\mathbf{E}\left(\left(Y^{(\alpha/2)}\right)^{\gamma}\right) = \frac{\Gamma(1-\frac{2\gamma}{\alpha})}{\Gamma(1-\gamma)},$$

and, further, from (47) for $-1 < \gamma < \alpha$

$$m_\gamma := \mathbf{E}\left(|X^{(\alpha)}|^{\gamma}\right) = 2^{\gamma}\,\Gamma(\frac{1+\gamma}{2})\,\Gamma(\frac{\alpha-\gamma}{\alpha})/\left(\sqrt{\pi}\,\Gamma(\frac{2-\gamma}{2})\right). \tag{48}$$

The constants with the associated reference numbers of the formulae where they appear in the paper are summarized in the following table.

| Constant | Value | Ref. |
|---|---|---|
| $c_1(\alpha)$ | $((\alpha-1)\,\pi\, m_{\alpha-1})\,/\varGamma(1/\alpha)$ | $(3), (56), (57)$ |
| $c_2(\alpha)$ | $(2(\alpha-1)\,m_{2(\alpha-1)})/(\alpha\, m_{\alpha-2})$ | $(5)$ |
| $c_3(\alpha,\gamma)$ | $(\gamma\, m_\gamma)/(\alpha\, m_{\gamma-\alpha})$ | $(7), (37)$ |
| $c_4(\alpha,\gamma)$ | $c_1(\alpha)/r(\alpha,\gamma)$ | $(41), (45)$ |
| $c_5(\alpha)$ | $\alpha/(2\,\varGamma(1-\alpha)\,\cos(\alpha\pi/2))$ | $(29)$ |
| $c_6(\alpha)$ | $(c_1(\alpha))^{-1} = (2\pi c_5(\alpha-1))^{-1}$ | $(30)$ |
| $c_7(\alpha)$ | $c_2(\alpha)\,(c_6(\alpha))^{\,2}$ | $(33)$ |
| $c_8(\alpha,\gamma)$ | $c_3(\alpha,2\gamma) - 2c_3(\alpha,\gamma)$ | $(38)$ |

We consider first the constant $c_3(\alpha,\gamma)$ and, for clarity, recall formula (36):

$$|X_t - x|^\gamma = |x|^\gamma + N_t^{(\gamma)} + A_t^{(\gamma)}, \tag{49}$$

with $\alpha - 1 < \gamma < \alpha$ and

$$A_t^{(\gamma)} = c_3(\alpha,\gamma)\int_0^t \frac{ds}{|X_s - x|^{\alpha-\gamma}}.$$

Notice that letting $\gamma \downarrow \alpha - 1$ yields, in a sense,

$$A_t^{(\alpha-1)} = c_1(\alpha)\, L_t^x, \tag{50}$$

although, using the value in the table, $c_3(\alpha,\gamma) \to 0$. From (49) it is seen that $f(y) = |y-x|^\gamma$ belongs to the domain of the extended generator $\mathcal{G}$, and, by scaling we obtain the following integral representation

$$c_3(\alpha,\gamma) = \int_{\mathbf{R}} \nu(dy)\,(|1+y|^\gamma - 1 - \gamma y)\,.$$

On the other hand, taking $x = 0$ in (49), and using scaling again together with (48), we get

$$\mathbf{E}\left(|X_t|^\gamma\right) = c_3(\alpha,\gamma)\int_0^t ds\, \mathbf{E}\left(|X_s|^{\gamma-\alpha}\right),$$

which is equivalent with

$$t^{\gamma/\alpha}\, m_\gamma = c_3(\alpha,\gamma)\,\frac{\alpha\, t^{\gamma/\alpha}}{\gamma}\, m_{\gamma-\alpha}$$

hence,

$$c_3(\alpha,\gamma) = \gamma\, m_\gamma/\alpha\, m_{\gamma-\alpha}.$$

A similar argument leads to an expression for $c_1(\alpha)$. From (50) we get

$$\mathbf{E}\left(|X_t|^{\alpha-1}\right) = c_1(\alpha)\,\mathbf{E}\left(L_t^0\right). \tag{51}$$

We derive from (51) the existence of a constant $c_0(\alpha)$ such that

$$\mathbf{E}\left(d_t L_t^0\right) = c_0(\alpha) dt\, t^{-1/\alpha},$$

and it follows from (50) that

$$m_{\alpha-1} = \alpha\, c_1(\alpha)\, c_0(\alpha)/(\alpha-1). \tag{52}$$

We now compute $c_0(\alpha)$ to obtain $c_1(\alpha)$ from (52). For this consider the identity (20) for $x = y = 0$

$$u^{(p)}(0) = \mathbf{E}_0\left(\int_0^\infty \mathrm{e}^{-p\,s} d_s L_s^0\right),$$

which in terms of $c_0(\alpha)$ reads

$$\frac{1}{\pi}\int_0^\infty \frac{d\xi}{p+\xi^\alpha} = c_0(\alpha)\int_0^\infty \mathrm{e}^{-p\,s} s^{-1/\alpha}\, ds.$$

An elementary computation reveals that

$$c_0(\alpha) = \frac{1}{\pi}\,\Gamma((\alpha+1)/\alpha),$$

hence,

$$c_1(\alpha) = \left((\alpha-1)\,\pi\, m_{\alpha-1}\right)/\Gamma(1/\alpha).$$

Next we find from formula (31) that

$$c_6(\alpha) = 1/c_1(\alpha). \tag{53}$$

To compute $c_8(\alpha,\gamma)$ for $\alpha-1 \le \gamma \le \alpha/2$ and the limiting case $c_2(\alpha) = c_8(\alpha,\alpha-1)$ notice from (39) that

$$c_8(\alpha,\gamma) = \int_{\mathbf{R}} \nu(dy)\left(|1+y|^\gamma - 1\right)^2.$$

Comparing the integral representations of $c_3$ and $c_8$ it is seen that

$$2c_3(\alpha,\gamma) + c_8(\alpha,\gamma) = c_3(\alpha, 2\gamma) \tag{54}$$

which can also be deduced from the following formulae

$$(i) \qquad \mathbf{E}\left(|X_t|^{2\,\gamma}\right) = 2\,\mathbf{E}\left(\int_0^t |X_s|^\gamma \, d_s A_s^{(\gamma)}\right) + \mathbf{E}\left(<N^{(\gamma)}>_t\right)$$
$$= \left(2c_3(\alpha,\gamma) + c_8(\alpha,\gamma)\right)\, \mathbf{E}\left(\int_0^t \frac{ds}{|X_s|^{\alpha-2\gamma}}\right),$$
$$(ii) \qquad \mathbf{E}\left(|X_t|^{2\,\gamma}\right) = c_3(\alpha, 2\gamma)\,\mathbf{E}\left(\int_0^t \frac{ds}{|X_s|^{\alpha-2\gamma}}\right).$$

The first one of these is an easy application of the Itô formula for semimartingales and the second one follows (49) because $\gamma \le \alpha/2$. From equation (54) we get

$$\begin{aligned} c_8(\alpha,\gamma) &= c_3(\alpha, 2\gamma) - 2c_3(\alpha,\gamma) \\ &= \frac{2\gamma}{\alpha}\left(\frac{m_{2\gamma}}{m_{2\gamma-\alpha}} - \frac{m_\gamma}{m_{\gamma-\alpha}}\right). \end{aligned}$$

The constant $c_2$ is now obtained by letting here $\gamma \to \alpha - 1$ and using $m_{-1} = +\infty$. Consequently

$$c_2(\alpha) = \frac{2(\alpha-1)}{\alpha}\,\frac{m_{2(\alpha-1)}}{m_{\alpha-2}}.$$

To find the constant $c_5(\alpha)$, we use the relationship (11) between $\Psi$ and $\nu$ which yields after substitution $y = \xi z$

$$c_5(\alpha) = \left(2\int_0^\infty \frac{1-\cos y}{y^{\alpha+1}}\,dy\right)^{-1}.$$

Integrating by parts and using the formulae 2.3.(1) p. 68 in Erdelyi et al. [7] lead us to the explicit value of the integral

$$\int_0^\infty \frac{1-\cos y}{y^{\alpha+1}}\,dy = \frac{\Gamma(1-\alpha)}{\alpha}\cos(\alpha\pi/2).$$

The constant $c_6(\alpha)$ can also clearly be expressed in terms of $c_5$

$$\begin{aligned} c_6(\alpha) &= \left(2\pi c_5(\alpha-1)\right)^{-1} = \frac{1}{\pi}\int_0^\infty \frac{1-\cos\xi}{\xi^{\,\alpha}}\,d\xi \\ &= \frac{1}{\pi}\,\frac{\Gamma(2-\alpha)}{\alpha-1}\cos((\alpha-1)\pi/2). \end{aligned}$$

It can be verified by the duplication formula for the Gamma function that this agrees with (53). It holds also that $c_6(\alpha) \to 1/2$ as $\alpha \uparrow 2$.

The constant $c_7$ is obtained by simply comparing the definitions of $N^x$ in (3) and $\tilde{N}^x$ in Proposition 3. We have

$$N_t^x = \frac{1}{c_6(\alpha)}\,\tilde{N}_t^x$$

implying

$$c_7(\alpha) = c_2(\alpha)\,(c_6(\alpha))^{\,2}.$$

# 6 Symmetric principal values of local times

Our previous results may be summarized as follows

1. for $\alpha - 1 \le \gamma < \alpha$ the process $\{|X_t - x|^\gamma\}$ is a submartingale whose Doob-Meyer decomposition is given by (36),
2. for $(\alpha-1)/2 < \gamma < \alpha - 1$ the process $\{|X_t - x|^\gamma\}$ is a Dirichlet process whose canonical decomposition is given by (40).

These results do not discuss whether $\{(X_t - x)^{\gamma,*}\}$, the symmetric power of order $\gamma$, i.e.,

$$(X_t - x)^{\gamma,*} := \operatorname{sgn}(X_t - x)\,|X_t - x|^\gamma, \tag{55}$$

is or is not a semimartingale or a Dirichlet process. In the present section it is seen that this question can be answered completely relying on some results in Fitzsimmons and Getoor [8] and [9], see also K. Yamada [19]. Let $x = 0$ in (55) and introduce the principal value integral (cf. (9))

$$\text{p.v.}\int_0^t \frac{ds}{X_s^{\theta,*}} := \int_0^\infty \frac{dz}{z^\theta}\left(L_t^z - L_t^{-z}\right),$$

where by the Hölder continuity (18) the integral is well defined for $\theta < (\alpha - 1)/2$.

**Proposition 6. a)** *For $\alpha-1 < \gamma < \alpha$ the process $\{X_t^{\gamma,*}\}$ is a semimartingale.*
**b)** *For $(\alpha-1)/2 < \gamma \le \alpha - 1$ the process $\{X_t^{\gamma,*}\}$ is a Dirichlet process and not a semimartingale.*
**c)** *In both cases the unique canonical decomposition of the process can be written as*

$$X_t^{\gamma,*}\, r^+(\alpha,\gamma) = N_t^{\gamma,*} + c_1(\alpha)\ \text{p.v.}\int_0^t \frac{ds}{X_s^{\alpha-\gamma,*}}, \tag{56}$$

*where*

$$r^+(\alpha,\gamma) = \int_0^\infty \frac{dx}{x^{\alpha-\gamma}}\left(|1-x|^{\alpha-1} - (1+x)^{\alpha-1}\right)$$

*and*

$$N_t^{\gamma,*} = \int_0^\infty \frac{dx}{x^{\alpha-\gamma}}\left(N_t^x - N_t^{-x}\right).$$

*In particular, for* $\gamma = \alpha - 1$

$$X_t^{\alpha-1,*}\, r^+(\alpha,1) = N_t^{\alpha-1,*} + c_1(\alpha) \text{ p.v.} \int_0^t \frac{ds}{X_s}. \tag{57}$$

*Proof.* Because $\{X_t\}$ is a martingale, it follows from the Ito formula for semimartingales (15) that for $1 \le \gamma \le \alpha$ the process $\{X_t^{\gamma,*}\}$ is a semimartingale. The other statements in a) and b) are derived from the decomposition (56) which we now verify similarly as (40) in Proposition 5. Hence, we start again from the identity (3) considered at $x$ and $-x$, and write, informally

$$\begin{aligned}&\int_0^\infty \frac{dx}{x^{\alpha-\gamma}} \left(|X_t - x|^{\alpha-1} - |X_t + x|^{\alpha-1}\right) \\ &\quad = \int_0^\infty \frac{dx}{x^{\alpha-\gamma}} \left(N_t^x - N_t^{-x}\right) + c_1(\alpha) \int_0^\infty \frac{dx}{x^{\alpha-\gamma}} \left(L_t^x - L_t^{-x}\right),\end{aligned} \tag{58}$$

To analyze the integral on the left hand side notice that

$$R(a;\alpha,\gamma) = \int_0^\infty \frac{dx}{x^{\alpha-\gamma}} \left(|a-x|^{\alpha-1} - |a+x|^{\alpha-1}\right).$$

is absolutely convergent and

$$R(a;\alpha,\gamma) = a^{\gamma,*}\, r^+(\alpha,\gamma).$$

Now the rest of the proof is very similar to that of Proposition 5 b), and is therefore omitted.

*Remark 3.* **a)** The increasing process associated with $N^{\gamma,*}$ is given by

$$\begin{aligned}<N_t^{\gamma,*}>_t &= (r^+(\alpha,\gamma))^2 \int_0^t ds \int_{\mathbf{R}} \nu(dz) \left((X_s+z)^{\alpha-\gamma,*} - X_s^{\alpha-\gamma,*}\right)^2 \\ &= (r^+(\alpha,\gamma))^2 \int_0^t \frac{ds}{|X_s|^{\alpha-2\gamma}} \int_{\mathbf{R}} \nu(dz) \left((1+z)^{\alpha-\gamma,*} - 1\right)^2.\end{aligned}$$

We also have by scaling

$$\begin{aligned}\mathbf{E}\left(\int_0^t \frac{ds}{|X_s|^{\alpha-2\gamma}}\right) &= \int_0^t s^{(2\gamma-\alpha)/\alpha}\, ds\, \mathbf{E}\left(|X_1|^{2\gamma-\alpha}\right) \\ &= \frac{\alpha}{2\gamma}\, t^{2\gamma/\alpha}\, \mathbf{E}\left(|X_1|^{2\gamma-\alpha}\right).\end{aligned}$$

**b)** Since

$$|X_t|^\gamma = (X_t^+)^\gamma + (X_t^-)^\gamma$$

and

$$|X_t|^{\gamma,*} = (X_t^+)^\gamma - (X_t^-)^\gamma$$

it is straightforward to derive the decomposition formulae for $\{(X_t^+)^\gamma\}$ and $\{(X_t^-)^\gamma\}$, and we leave this to the reader.
**c)** Note how different (57) is in the Brownian case $\alpha = 2$, for which on one hand $\{B_t\}$ is a martingale, and on the other hand

$$\varphi(B_t) = \int_0^t \log|B_s|\, dB_s + \frac{1}{2}\ \text{p.v.} \int_0^t \frac{ds}{B_s}$$

with $\varphi(x) = x\log|x| - x$. For principal values of Brownian motion and extensions of Itô's formula, see Yor [22], [23] and Cherny [5].

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