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On Russian Options

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Abstract. In this paper we compute the hedging portfolio for the Russian option as introduced by Shepp and Shiryayev (1993). It is also seen that the derivation of the valuation formula in Shepp and Shiryayev (1993), (1994) can be shortened by making use of a generalization of Lévy's theorem for a Brownian motion with drift. The associated optimal stopping problem is solved with the technique exploiting the represention theory of excessive functions as presented in Salminen (1985). We conclude by discussing valuation of some related contingent claims (or payment functionals).

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1. Introduction. Let $S = \{S_t : \ t \geq 0\}$ be a geometrical Brownian motion started at x with parameters μ and σ, GBM(μ, σ^2). Then S can be realized by defining

$$S_t := x \ \exp((\mu - \sigma^2/2)t + \sigma W_t), \quad t \geq 0,$$

where W is a standard Brownian motion, BM. We view S as the price process of a stock, and assume that there exists also a riskless alternative to invest, e.g., on bonds. Let $r > 0$ be the interest rate of the bonds. Then the price process of a bond with the initial price y is $B = \{B_t : \ t \geq 0\}$, where

$$B_t = y \, e^{r\,t}.$$

The Russian option as introduced by Shepp and Shiryayev (1993) is a perpetual American contingent claim, ACC, with the payment functional (or process)

$$t \mapsto G_t := e^{-\lambda t} \max\{\sup_{s\leq t} S_s, \beta x\}. \tag{1}$$

where $\lambda > 0$, $\beta \geq 1$, and $S_0 = x$. In other words, the buyer of the Russian option can choose a (random) time point τ to excercise the option at which time point she/he receives the payment G_τ from the seller of the option.

The solution to the pricing problem of the Russian option given in Shepp and Shiryayev (1993) is considerably simplified in Shepp and Shiryayev (1994) by using a trick which transforms the original 2-dimensional problem to a 1-dimensional one. In economical terms, the trick is to use the stock price as the numeraire instead of the bond price. This method is studied more generally in Shiryayev et al. (1994), and the term "dual martingale measure" is introduced to describe it. Moreover, in Kramkov and Mordecky (1994) it is shown that the method can also be applied to price the American option with the payment process

$$t \mapsto e^{-\lambda t}(\int_0^t S_s\, ds + \beta x),$$

where $\lambda > 0$, $\beta \geq 1$, and $S_0 = x$.

In this note Section 2 we present a new approach to dual martingale measure based on excessive transforms (or Doob's h–transforms). It is also seen that making use of the following theorem which for BM is due to Lévy offers a shortcut to the result of Shepp and Shiryayev (1994) on the Russian option.

Theorem 1. Let $W^{(c)}$ be a Brownian motion with drift c, BM(c). Then the process $Z^\star$ defined by

$$Z^\star_t := \sup_{s\leq t} W^{(c)}_s - W^{(c)}_t, \quad Z^\star_0 := 0$$

is a BM($-c$) reflected at 0 and living in $[0,\infty)$, RBM($-c$). Moreover, the filtration generated by $Z^\star$ is the same as the filtration generated by $W^{(c)}$.

For a proof see Harrison (1990) p. 49-50, 81. The claim concerning the filtrations follows from Skorokhod's reflection equation.

To find the rational price of an American contingent claim (with infinite horizon) is a problem in optimal stopping (see Karatzas (1988), Myneni (1992), and Shiryayev et.al. (1994)). If the payment or reward function only depends on the present state of the underlying process the optimal stopping problem can be solved by exploiting the properties of excessive functions via their representations in terms of the minimal excessive functions (see Salminen (1985)).

This approach also explains why the often used condition of smooth-fit must hold.

In Section 4 we compute the hedging portfolio process associated to the Russian option. It is seen that both the amount invested in the stock and the bond are non-negative. In other words, the optimal strategy has the appealing property that there is no short selling or borrowing.

We conclude the paper by indicating how to price American contingent claims with the payment process of the type

$$t \mapsto e^{-\lambda t} S_t^a \sup_{s \le t} S_s^b,$$

where $\lambda \geq 0$ and a and b are arbitrary real numbers.

The main facts on the Russian options can be presented in a compressed form as follows:

- The rational price of the option is $xV^\star(\beta)$ where x is the initial prize of the stock and $V^\star(\beta) > 1$ is independent of x. To give some numerical flavour, let $\beta = 1,\ \lambda = 0.125,\ r = 0.025$, and $\sigma = 0.3$ then $V^\star = 1.174$.

- The rational time to exercise the Russian option is when the difference of $\max\{\sup_{s \le t} S_s, \beta x\}$ and the present value S_t compared to S_t exceeds a threshold depending on the parameters of the model. In our numerical example the threshold value in precentages is 36.5%.

- Letting $\{X_t^\star : t \geq 0\}$ be the optimal wealth process and $\pi_t^\star$ the optimal amount invested in the stock at time t (in the case $\beta = 1$) then we have

$$X_t^\star = e^{-\lambda t} S_t\, V^\star(\sup_{s \le t} S_s / S_t) =: e^{-\lambda t} F(S_t, \sup_{s \le t} S_s).$$

and

$$\pi_t^\star = S_t\, F_1'(S_t, \sup_{s \le t} S_s).$$

Moreover, $0 \leq \pi_t^\star \leq X_t^\star$ for all $t \geq 0$ and $\pi_t^\star = X_t^\star$ if and only if $S_t = \sup_{s \le t} S_s$.

2. Martingale measures. We introduce the martingale measure and the dual martingale measure in the canonical space $(\mathbf{C}, \mathcal{C}, \{\mathcal{C}_t\})$ where $\mathbf{C}$ consists of all continuous functions $\omega : [0, \infty) \mapsto \mathbf{R}$, $\mathcal{C}_t = \sigma\{\omega(s) : \ s \leq t\}$ is the smallest σ–algebra making coordinate mappings up to time t measurable, and $\mathcal{C} = \sigma\{\omega(s) : \ s \geq 0\}$.

Definition. (i) For given $\sigma > 0$ and $r > 0$ the probability measure $\mathbf{P}^+$ in $(\mathbf{C}, \mathcal{C}, \{\mathcal{C}_t\})$ such that the process $\{e^{-rt}\omega(t) : \ t \geq 0\}$ is a $\mathrm{GBM}(0, \sigma^2)$ started at 1 is called the martingale measure.

(ii) For given $\sigma > 0$ and $r > 0$ the probability measure $\mathbf{P}^-$ in $(\mathbf{C}, \mathcal{C}, \{\mathcal{C}_t\})$ such that the process $\{e^{-rt}\omega(t) : t \geq 0\}$ is a GBM(σ^2, σ^2) started at 1 is called the dual martingale measure.

From the first part of the definition we have the interpretation that under $\mathbf{P}^+$ the stock price process when the bond price is used as the numeraire is a martingale. Below we use the notation $\mathbf{P}_x^+$ ($\mathbf{P}_x^-$) for the measure of GBM(r, σ^2) (GBM$(r + \sigma^2, \sigma^2)$) started at $x > 0$.

Proposition 1. Under $\mathbf{P}^-$ the process $\{e^{rt}/\omega(t) : t \geq 0\}$ is a GBM$(0, \sigma^2)$ started at 1, that is, under $\mathbf{P}^-$ the bond price process when the stock price is used as the numeraire is a martingale.

Proof. By the definition

$$(\{e^{rt}/\omega(t) : \ t \geq 0\}, \mathbf{P}^-) \ \sim \ (\{e^{rt}e^{-(r+\sigma^2/2)t-\sigma W_t} : \ t \geq 0\}, \mathbf{P}_0)$$

where $(W, \mathbf{P}_0)$ is a standard BM started at 0 and $\sim$ means that the processes are identical in law. Consequently,

$$(\{e^{rt}/\omega(t) : \ t \geq 0\}, \mathbf{P}^-) \ \sim \ (\{e^{-\sigma^2 t/2-\sigma W_t} : \ t \geq 0\}, \mathbf{P}_0).$$

Because $-W \ \sim \ W$ the proof is complete. //

Remark. See Shiryayev et.al. (1994) and Shiryayev and Shepp (1994) for a construction of the dual martingale measure via stochastic calculus.

Proposition 2. For every $t > 0$ and $x > 0$ the measures $\mathbf{P}_x^+$ and $\mathbf{P}_x^-$ when restricted to $\mathcal{C}_t$ are equivalent, and

$$\left.\frac{d\mathbf{P}_x^+}{d\mathbf{P}_x^-}\right|_{\mathcal{C}_t} = \frac{xe^{rt}}{\omega(t)} \qquad \mathbf{P}_x^- \text{-a.s.}$$

Moreover, for a stopping time τ let $\mathcal{C}_\tau$ be the σ–algebra containing events "observed" before τ. Then

$$\left.\frac{d\mathbf{P}_x^+}{d\mathbf{P}_x^-}\right|_{\mathcal{C}_\tau} = \frac{xe^{r\tau}}{\omega(\tau)} \qquad \mathbf{P}_x^- \text{-a.s. on the set } \{\tau < \infty\}.$$

Proof. Let X be a GBM$(0,\sigma^2)$. Then X is a 1-dimensional diffusion and we may take $s(x) = x$ to be its scale function. Let $X^\uparrow$ be the Doob's h-transform of X obtained by using s as h, that is, $X^\uparrow$ is the diffusion with the semigroup

$$\mathbf{E}_x^\uparrow(F(\omega(t)) := \frac{1}{x}\mathbf{E}_x^\downarrow(\omega(t)F(\omega(t))),$$

where F is a bounded and measurable function and $\mathbf{E}^\downarrow$ is the expectation operator associated to X. Straightforward calculations (cf. Borodin and Salminen (1996) II.30) show that $X^\uparrow$ is a GBM(σ^2,σ^2). Conversely, because $s^\uparrow(x) = -1/x$ is the scale function of $X^\uparrow$ the h-transform of $X^\uparrow$ using $-s^\uparrow$ as h is identical in law with X. Therefore,

$$\mathbf{E}_x^\downarrow(F(\omega(t)) = x\mathbf{E}_x^\uparrow(F(\omega(t))/\omega(t)).$$

Combining this with the definitions of $\mathbf{P}^+$ and $\mathbf{P}^-$ prove the claim at a fixed time t. To extend the result for stopping times is a standard truncation argument, and the details are omitted. //

Remark. The notations above with $\uparrow$ and $\downarrow$ (borrowed from Pitman and Yor (1981)) refer to the fact that $X^\uparrow$ may be viewed as X conditioned to have $\lim X_t = \infty$ a.s. and X as $X^\uparrow$ conditioned to have $\lim X_t^\uparrow = 0$ a.s.

3. Pricing Russian options. As pointed out in Introduction to find the fair price of an ACC is a problem in optimal stopping. The basic result is that the *fair* or *rational price*, denoted V, does not depend on the drift parameter μ of the modell, and is given by

$$V(x) = \sup_\tau \mathbf{E}_x^+(e^{-r\tau}Y_\tau(\omega))$$

where $t \mapsto Y_t$ is the payment process assumed to be continuous and $\mathcal{C}_t-$adapted, the supremum is taken with respect to all stopping times τ in the natural filtration $\{\mathcal{C}_t\}$ of the co-ordinate process ω (interpreted as the stock price process S). We refer to Shiryayev et. al (1994) p. 89 for a discussion about bounded, finite and general stopping times in this context. Recall that we set

$$Y_{\tau(\omega)}(\omega) = \limsup_{t\to\infty} Y_t(\omega)$$

in the case $\tau(\omega) = \infty$. A *rational exercise time* of the option is any finite stopping time realizing the supremum V. If the condition

$$\mathbf{E}^+(\sup_{t\geq 0} e^{-rt}Y_t) < \infty. \tag{2}$$

holds it is known (see Karatzas (1991) p. 25-26) that a rational exercise time exists and has a simple form in terms of the wealth process.

The technical condition (2) is checked in the next proposition for the Russian option. We remark, however, that this is not really needed for further developments because it is possible to find the rational exercise time and the corresponding wealth process explicitly, see Section 4.

Proposition 3. The condition (2) holds for the Russian option, i.e.,

$$\mathbf{E}^+(\sup_{t\geq 0} e^{-rt}\, G_t) < \infty\ ,$$

where G is as given in (1).

Proof. Consider

$$\begin{aligned}
\mathbf{E}^+(\sup_{t\geq 0} e^{-(r+\lambda)t} \sup_{s\leq t} \omega_s) &= \mathbf{E}^+(\sup_{t\geq 0} e^{-(r+\lambda)t} \sup_{s\leq t}(e^{-(r+\lambda)s} e^{(r+\lambda)s}\omega_s)) \\
&\leq \mathbf{E}^+(\sup_{t\geq 0}\sup_{s\leq t}(e^{-(r+\lambda)s}\omega_s)) \\
&= \mathbf{E}^+(\sup_{t\geq 0} e^{-(r+\lambda)t}\omega_t).
\end{aligned}$$

Under the measure $\mathbf{P}^+$ the process $D := \{e^{-(r+\lambda)t}\omega_t \ :\ t\geq 0\}$ is a GBM$(-\lambda,\sigma^2)$. Let $\mathbf{P}^D_y$ denote the law of D when started at y and $\tau_x := \inf\{t\ :\ D_t = x\}$. Then we have for $y < x$

$$\mathbf{P}^+_y(\sup_{t\geq 0} e^{-(r+\lambda)t}\omega_t > x) = \mathbf{P}^D_y(\tau_x < \infty) = (\frac{y}{x})^{1-p}$$

where $p = -2\lambda/\sigma^2 < 0$ is the parameter of GBM$(-\lambda,\sigma^2)$ (see Borodin and Salminen (1996) p. 113). Because the function $x \mapsto x^{p-1}$ is integrable on (y,∞) for any $y > 0$ the proof is complete. //

Consider now

$$\begin{aligned}
&\mathbf{E}^+_x(e^{-r\tau}G_\tau(\omega)\mathbf{1}_{\{\tau<\infty\}}) \\
=\ &\mathbf{E}^+_x(e^{-(r+\lambda)\tau}\mathbf{1}_{\{\tau<\tau_{\beta x}\}}\beta x) + \mathbf{E}^+_x(e^{-(r+\lambda)\tau}\mathbf{1}_{\{\tau_{\beta x}<\tau<\infty\}}\sup_{s\leq\tau}\omega(s)),
\end{aligned}$$

where τ is an arbitrary stopping time and $\tau_{\beta x} := \inf\{t :\ \omega(t) = \beta x\}$. Changing to the dual martingale measure gives

$$\begin{aligned}
&\mathbf{E}^+_x(e^{-r\tau}G_\tau(\omega)\mathbf{1}_{\{\tau<\infty\}}) \\
&= x\mathbf{E}^-_x(e^{-(r+\lambda)\tau}\mathbf{1}_{\{\tau<\tau_{\beta x}\}}\ \frac{\beta x}{e^{-r\tau}\omega(\tau)}) + x\mathbf{E}^-_x(e^{-(r+\lambda)\tau}\mathbf{1}_{\{\tau_{\beta x}<\tau<\infty\}}\frac{\sup_{s\leq\tau}\omega(s)}{e^{-r\tau}\omega(\tau)}).
\end{aligned}$$

We proceed with the following lemma where the notation τ_z is used for the first hitting time of z (in the canonical framework).

Lemma 1. Let $\delta = (r+\sigma^2/2)/\sigma$ and $\mathbf{P}^\delta_x$ be the measure associated to BM(δ). Then the $\mathbf{P}^-_x$-distribution of $\tau_{\beta x}$ is the same as the $\mathbf{P}^\delta_{-\hat\beta}$-distribution of τ_0 with $\hat\beta = \ln\beta/\sigma$.

Proof. Under $\mathbf{P}^-$ the coordinate process is GBM$(r+\sigma^2, \sigma^2)$. Hence,

$$\begin{aligned}
(\tau_{\beta x}, \mathbf{P}^-_x) &\sim (\inf\{t:\ x e^{\sigma\omega(t)+(r+\sigma^2/2)t} = \beta x\}, \mathbf{P}_0)\\
&\sim (\hat\tau := \inf\{t:\ \omega(t)+\delta t = \hat\beta\}, \mathbf{P}_0),\\
&\sim (\tau_{\hat\beta}, \mathbf{P}^\delta_0),\\
&\sim (\tau_0, \mathbf{P}^\delta_{-\hat\beta}). \quad //
\end{aligned}$$

From Lemma 1 it follows that for a fixed time t

$$\begin{aligned}
x\mathbf{E}^-_x(e^{-(r+\lambda)t}\mathbf{1}_{\{t<\tau_{\beta x}\}}\frac{\beta x}{e^{-rt}\omega(t)}) &= \beta x\mathbf{E}_0(e^{-\lambda t}\, e^{-\sigma\omega(t)-(r+\sigma^2/2)t}\mathbf{1}_{\{t<\hat\tau\}})\\
&= \beta x\mathbf{E}^\delta_{-\hat\beta}(e^{-\lambda t-\sigma(\omega(t)+\hat\beta)}\mathbf{1}_{\{t<\tau_0\}})\\
&= x\mathbf{E}^\delta_{-\hat\beta}(e^{-\lambda t-\sigma\omega(t)}\mathbf{1}_{\{t<\tau_0\}}),
\end{aligned}$$

and

$$\begin{aligned}
x\mathbf{E}^-_x(e^{-(r+\lambda)t}\mathbf{1}_{\{\tau_{\beta x}<t<\infty\}}\frac{\sup_{s\le t}\omega(s)}{e^{-rt}\omega(t)}) &= x\mathbf{E}^\delta_0(e^{-\lambda t+\sigma(\sup_{s\le t}\omega(s)-\omega(t))}\mathbf{1}_{\{\tau_{\hat\beta}<t<\infty\}})\\
&= x\mathbf{E}^\delta_{-\hat\beta}(e^{-\lambda t+\sigma(\sup_{s\le t}\omega(s)-\omega(t))}\mathbf{1}_{\{\tau_0<t<\infty\}}).
\end{aligned}$$

Consequently, for every fixed time t we have by Theorem 1

$$\mathbf{E}^+_x(e^{-rt}G_t(\omega)) = x\,\mathbf{E}^\circ_{-\hat\beta}(e^{-\lambda t}e^{\sigma|\omega(t)|}),$$

where $\mathbf{P}^\circ$ governs a process which on $(-\infty, 0)$ behaves like BM(δ) and on $[0, +\infty)$ like RBM$(-\delta)$. By symmetry we also have

$$\mathbf{E}^+_x(e^{-rt}G_t(\omega)) = x\,\mathbf{E}^\star_{\hat\beta}(e^{-\lambda t}e^{\sigma\omega(t)}), \tag{3}$$

where $\mathbf{P}^\star$ is the measure associated to RBM$(-\delta)$. Next we solve the optimal stopping problem suggested by the right hand side of (3).

Theorem 2. For λ, r and σ in $\mathbf{R}_+$ let $\gamma = \sqrt{2\lambda + \delta^2}$ and $\delta = (r + \sigma^2/2)/\sigma$, and introduce for $x > 0$ the function

$$x \mapsto \psi^\star(x) = \frac{\gamma - \delta}{2\gamma} e^{(\gamma+\delta)x} + \frac{\gamma + \delta}{2\gamma} e^{-(\gamma-\delta)x}.$$

The optimal stopping problem

$$\sup_\tau \mathbf{E}^\star_{\hat\beta}(e^{-\lambda\tau + \sigma\omega(\tau)}) =: V^\star(\beta)$$

has a solution if and only if $\lambda > 0$ and an optimal time in this case is

$$\tau_{a^\star} = \inf\{t : \, \omega(t) \geq a^\star \},$$

where $\hat\beta = \ln\beta/\sigma$ and $a^\star$ is the unique positive solution of the equation

$$\frac{d}{dx}\psi^\star(x) - \sigma\psi^\star(x) = 0,$$

that is

$$a^\star = \frac{1}{2\gamma} \ln\Big((1 + \frac{\sigma}{\gamma - \delta})/(1 - \frac{\sigma}{\gamma + \delta})\Big)$$

Moreover, $\tau_{a^\star} < \infty$ $\mathbf{P}^\star_{\hat\beta}$–a.s. and

$$V^\star(\beta) = \begin{cases} \beta\, e^{\sigma(a^\star - \hat\beta)} \, \dfrac{\psi^\star(\hat\beta)}{\psi^\star(a^\star)}, & \hat\beta < a^\star, \\ \beta, & \text{otherwise.} \end{cases}$$

Proof is given in Appendix.

The solution to the pricing problem of the Russian option is now obtained from Theorem 2 and given in the following

Theorem 3. The rational price of the Russian option is

$$V(x, \beta) = \sup_\tau \mathbf{E}^+_x(e^{-r\tau} G^{(1)}_\tau(\omega)) = xV^\star(\beta),$$

and the rational time to exercise the option is

$$\tau^\star := \inf\{t : \; \max\{\sup_{s \leq t} \omega(s), \, \beta x\} \geq e^{\sigma a^\star} \omega(t)\}.$$

Proof. Noting that $\beta > 1$ enters into the formulas as the initial state it is enough to consider only the case $\beta = 1$. Because the filtrations generated by $t \mapsto \sup_{s\le t}\omega(s) - \omega(t)$ and $t \mapsto \omega(t)$ are the same by the Skorokhod reflection equation the solution of the optimal stopping problem in Theorem 2 gives also the solution of the problem

$$\sup_{\tau} \mathbf{E}^{\delta}_{\hat\beta}(e^{-\lambda\tau+\sigma(\sup_{s\le\tau}\omega(s)-\omega(\tau))})$$

the optimal stopping time being

$$\tau^{\circ} = \inf\{t : \sup_{s\le t}\omega(s) - \omega(t) \ge a^{\star}\}.$$

Because the transformation from BM to GBM is one-to-one the filtrations they generate are the same and the optimal stopping time τ° transforms to $\tau^{\star}$ for the original price process (under $\mathbf{P}^{-}$ and $\mathbf{P}^{+}$). Further, because $\tau_{a^\star} < \infty$ $\mathbf{P}^{\star}_0$–a.s. it follows that $\tau^\star < \infty$ $\mathbf{P}^{+}_x$–a.s. and from (3)

$$x\,V^{\star}(1) = x\mathbf{E}^{\star}_0(e^{-\lambda\tau_{a^\star}+\sigma\,a^\star}) = \mathbf{E}^{+}_x(e^{-r\tau^\star}G_{\tau^\star}(\omega)) \quad //$$

Remarks. **(i)** The results in Theorems 1 and 2 are due to Shepp and Shiryayev (1993) and (1994). Our formulation is, however, different and it demands some elementary calculations to show that it equals with the solution presented in Shepp and Shiryayev (1993). We leave the details to the reader but display the expressions of Shepp and Shiryayev for use in the next section: let

$$x_1 = -\frac{\gamma-\delta}{\sigma}, \quad x_2 = \frac{\gamma+\delta}{\sigma} \quad \text{and} \quad c_\star := e^{\sigma a^\star} = \Big(\frac{x_2(x_1-1)}{x_1(x_2-1)}\Big)^{1/x_2-x_1}$$

then

$$V^{\star}(\beta) = \begin{cases} c_\star \dfrac{x_2\beta^{x_1} - x_1\beta^{x_2}}{x_2c_\star^{x_1} - x_1c_\star^{x_2}}, & \beta < c_\star, \\ \beta, & \text{otherwise.} \end{cases}$$

(ii) Shepp and Shiryayev (1994) study directly under the dual martingale measure the process

$$(t,\omega) \mapsto \frac{\sup_{s\le t}\omega(s)}{\omega(t)}$$

and prove that it is Markov with respect to the filtration $\{\mathcal{C}_t\}$. This is a key point. To higlight the problem slightly more consider the process

$$(t,\omega) \mapsto \frac{\sup_{s\le t}\omega^2(s)}{\omega(t)}.$$

By a theorem by Pitman and Rogers (1981) this is a Markov process in its *own* filtration which, however, is strictly smaller than the Brownian filtration (see Revuz and Yor (1991)). Consequently, we cannot proceed as above to price an ACC with the payment process

$$(t,\omega) \mapsto \sup_{s\le t}\omega^2(s).$$

(iii) One interpretation of the parameter λ in our models is via dividend payments. Indeed, assume that the stock pays continuously dividends with rate ρ, that is, the dividend yield process D is given by $dD_t = \rho S_t dt$. For details of the model with dividends see, e.g., Wilmott et al. (1996) p. 90. As pointed out in Duffie and Harrison [2], see also Shepp and Shiryayev (1994), from the result without dividends we get the result with dividends by changing λ to $\lambda+\rho$ and r to $r-\rho$.

4. Hedging the Russian option. The seller of an option must create, using the money she/he obtains from the buyer, so much wealth that she/he can cover the claim of the buyer at any future time point. Borrowing and short selling are allowed. It is assumed that the *wealth process*, $\{X_t : \ t\ge 0\}$, of the seller obeys the stochastic differential equation

$$dX_t = \pi_t\frac{dS_t}{S_t} + (X_t-\pi_t)\frac{dB_t}{B_t} - dC_t, \tag{4}$$

where π_t is the amount invested in stock at time t and $C := \{C_t \ : \ t\ge 0\}$ is the *consumption process*. The *portfolio process* $\pi := \{\pi_t \ : \ t \ge 0\}$ is supposed to be progressively measurable such that for all $t\ge 0$ a.s.

$$\int_0^t |\pi_s|ds < \infty.$$

The consumption process C is assumed to be non-decreasing, right continuous, $\mathcal{C}_t-$adapted and starting at 0. The solution of (4) satisfies

$$e^{-rt}X_t - X_0 = (\mu-r)\int_0^t e^{-rs}\pi_s ds + \sigma\int_0^t e^{-rs}\pi_s dW_s - \int_0^t e^{-rs}dC_s.$$

By the fundamental pricing theorem (see Karatzas (1991) p. 26) of perpetual ACC's there exist a (*tame*) portfolio process $\pi^\star$ and a consumption process $C^\star$ such that if the initial capital is

$$v_o := V(x) = \sup_\tau \mathbf{E}_x^+(e^{-r\tau}G_\tau(\omega))$$

then the corresponding wealth process $X^\star$ is given by

$$X_t^\star = \operatorname{ess\,sup}_{\tau\in M_t}\mathbf{E}^+(e^{-r(\tau-t)}\,G_\tau\,|\,\mathcal{C}_t),$$

where M_t is the class of stopping times taking values bigger than t. Moreover, it holds that $X_t^\star \geq G_t$ for all $t\geq 0$, and

$$\tau^\star := \inf\{t\,:\,X_t^\star = G_t\}$$

is the rational (or optimal) exercise time of the Russian option.

We compute now the processes $X^\star$ and $\pi^\star$ for the Russian option in the case $\beta = 1$. The case $\beta > 1$ is only notationally more complicated. Let $\tau\in M_t$ and consider

$$\begin{aligned}
&\mathbf{E}^+(e^{-r(\tau-t)}e^{-\lambda\tau}\sup_{s\leq\tau}\omega(s)\,|\,\mathcal{C}_t)\\
=\;& e^{-\lambda t}\mathbf{E}^+(e^{-(r+\lambda)(\tau-t)}\max(\sup_{s\leq t}\omega(s),\sup_{0\leq s\leq\tau-t}\omega(s)\circ\theta_t)\mid\mathcal{C}_t)\\
=\;& e^{-\lambda t}\mathbf{E}^+(e^{-(r+\lambda)(\tau-t)}\max(\beta_t\,\omega(t),\sup_{0\leq s\leq\tau-t}\omega(s)\circ\theta_t)\mid\mathcal{C}_t)
\end{aligned}$$

where $\beta_t = \sup_{s\leq t}\omega(s)/\omega(t)$ and θ_t is the usual shift operator. Taking ess sup over M_t gives (a.s.)

$$\begin{aligned}
X_t^\star &= e^{-\lambda t}V(\omega(t),\beta_t) = e^{-\lambda t}\omega(t)V^\star(\beta_t)\\
&=: e^{-\lambda t}F(\omega(t),\sup_{s\leq t}\omega(s))
\end{aligned}$$

with the notation as given in Theorem 3. Using the expression for V from Remark (i) in Section 3 and introducing

$$C(c_\star) = \frac{c_\star}{x_2c_\star^{x_1} - x_1c_\star^{x_2}}$$

the wealth process can be written as

$$X_t^\star = \begin{cases} C(c_\star)e^{-\lambda t}\omega(t)\Big(x_2\Big(\dfrac{\sup\omega(s)}{\omega(t)}\Big)^{x_1} - x_1\Big(\dfrac{\sup\omega(s)}{\omega(t)}\Big)^{x_2}\Big), & \text{if } \sup_{s\leq t}\omega(s) < c_\star\omega(t),\\ e^{-\lambda t}\sup_{s\leq t}\omega(s), & \text{if } \sup_{s\leq t}\omega(s)\geq c_\star\omega(t).\end{cases}$$

The amount invested in the stock at time t is obtained from the coefficient of the martingale part of the wealth process. Hence, by Ito's formula we have

$$\begin{aligned}
\pi_t^\star &= \omega(t)\, F_1'(\omega(t), \sup_{s\le t}\omega(s)) \\
&= C(c_\star)e^{-\lambda t}\omega(t)\Big(x_2(1-x_1)\Big(\frac{\sup\omega(s)}{\omega(t)}\Big)^{x_1} - x_1(1-x_2)\Big(\frac{\sup\omega(s)}{\omega(t)}\Big)^{x_2}\Big) \\
&= X_t^\star + C(c_\star)e^{-\lambda t}\omega(t)\, x_1x_2\Big(\Big(\frac{\sup\omega(s)}{\omega(t)}\Big)^{x_2} - \Big(\frac{\sup\omega(s)}{\omega(t)}\Big)^{x_1}\Big).
\end{aligned}$$

It follows that $\pi_t^\star = X_t^\star$ when $\omega(t) = \sup_{s\le t}\omega(s)$, i.e., at these moments all the wealth of the seller is in the stock. In particular, $\pi_0^\star = X_0^\star = v_o$. Further, $X_t^\star \ge \pi_t^\star$ for all t because $x_1 < 0 < x_2$. Next, consider the amount invested in the stock at time $\tau^\star$. Because $X_{\tau^\star}^\star = e^{-\lambda\tau^\star}\sup_{s\le\tau^\star}\omega(s)$ we have

$$\pi_{\tau^\star}^\star = X_{\tau\star}^\star\big(1 + \frac{C(c_\star)}{c_\star}x_1x_2(c_\star^{x_2} - c_\star^{x_1})\big)$$

and using the definition of $c_\star$ it is seen that $\pi_{\tau^\star}^\star = 0$. From these considerations it follows that the optimal hedging portfolio is constructed without short selling or borrowing. The consumption process (or the cash-flow process) is proportional to the time $X^\star$ spends below the level $\sup_{s\le t}\omega(s)/c_\star$ (note that the level changes when the new maximum is attained). In other words, the seller can consume the interest the amount $X_{\tau^\star}^\star$ pays during the time the stock price is below the level $\sup_{s\le t}\omega(s)/c_\star$.

5. Related contingent claims. Consider now an ACC with the payment process

$$t \mapsto e^{-\lambda t}S_t^a \sup_{s\le t} S_s^b, \tag{5}$$

where a and $b \ne 0$ are arbitrary real numbers. These can be analyzed by using the multiplicative property of GBM, i.e., if S is a GBM(μ,σ^2) then S^b is a GBM$(\mu_b, b^2\sigma^2)$ with

$$\mu_b = b\mu - \frac{\sigma^2}{2}b(1-b).$$

First let $a = 0$. For a fixed time t and in the canonical setting we have

$$\mathbf{E}_x^+(e^{-(r+\lambda)t}\sup_{s\le t}\omega(s)^b) = \mathbf{E}_{x^b}^{+,b}(e^{-(r+\lambda)t}\sup_{s\le t}\omega(s)),$$

where $\mathbf{P}^{+,b}$ governs a GBM$(r_b, b^2\sigma^2)$. The associated optimal stopping problem can be solved using the dual martingale measure approach as presented in

Section 3 – only the parameters are different. In particular, it is seen that the problem has a solution if (and only if)

$$\lambda_b := \lambda + r - r_b = \lambda + r - br + \frac{\sigma^2}{2} b(1-b) > 0.$$

Next let $a = 1$. Then

$$\begin{aligned} \mathbf{E}_x^+(e^{-(r+\lambda)t}\omega(t) \sup_{s \le t} \omega(s)^b) &= x\mathbf{E}_x^-(e^{-\lambda t} \sup_{s\le t} \omega(s)^b) \\ &= x\mathbf{E}_{x^b}^{-,b}(e^{-\lambda t} \sup_{s\le t} \omega(s)), \end{aligned}$$

where $\mathbf{P}^{-,b}$ governs a GBM$(\hat{r}_b, b^2\sigma^2)$ with

$$\hat{r}_b = b(r + \sigma^2) - \frac{\sigma^2}{2} b(1-b).$$

The associated optimal stopping problem has a solution if $\lambda > \hat{r}_b$.

Finally, let a be arbitrary but $\neq 0$ or 1. In this case we have

$$\mathbf{E}_x^+(e^{-(r+\lambda)t}\omega(t)^a \sup_{s\le t} \omega(s)^b) = \mathbf{E}_{x^a}^{+,a}(e^{-(r+\lambda)t}\omega(t) \sup_{s\le t} \omega(s)^{b/a}),$$

and the problem is in the form "$a = 1$".

Notice that because the problem with the payment process (5) can be transformed to the problem with the payment process (1) (with $\beta = 1$) the optimal stopping time is always of the form $\inf\{t : \sup_{s\le t} S_s \ge c\, S_t\}$. We leave it to the reader to compute the hedging portfolio process for the payment process (5).

6. Appendix. Here we prove Theorem 2 presented in Section 3. To start with we recall some general facts from the theory of optimal stopping. The function

$$V^\star(x) := \sup_\tau \mathbf{E}_x^\star(e^{-\lambda\tau + \sigma\omega(\tau)}).$$

is called the value function and $x \mapsto r(x) := e^{\sigma x}$ the reward function. Because the reward function is non-negative and continous the value $V^\star$ is the smallest λ–excessive majorant of r (see e.g. Shiryayev (1978) p. 118, 124). Moreover, letting $\Gamma_o := \{x : \; V^\star(x) = r(x)\}$ the optimal stopping time is $\tau_o := \inf\{t : \omega(t) \in \Gamma_o\}$ (Shiryayev (1978) p. 127).

We construct now $V^\star$ using the representation theory of excessive functions. This theory is usually called the Martin boundary theory. In Kunita and Watanabe (1965) Hunt processes are treated, and this is clearly enough for our case.

Let $Z^\star$ be a RBM$(-\delta)$ on $[0,\infty)$. To formulate the representation theorem we need the Green function of $Z^\star$. This is given, e.g., in Borodin and Salminen (1996) p. 110 in the following form

$$g^\star(x,y) = \frac{\psi^\star_\lambda(x)\varphi^\star_\lambda(y)}{w^\star_\lambda}, \quad 0 \le x \le y,$$

where $\varphi^\star_\lambda(y) = e^{-(\gamma-c)y}$, $\psi^\star_\lambda = \psi^\star$ with $\psi^\star$ as in Theorem 2, and $w^\star_\lambda$ is the Wronskian. To simplify the notation we will omit the subindex λ. Applying Theorem 3 p. 509 and Theorem 4 p. 513 in Kunita and Watanabe (1965) for $Z^\star$ (see Salminen (1985) for the explicit form, given below, of the representing measure) give us

Theorem 4. **(i)** The Martin compactification of the state space $I = [0,\infty)$ of $Z^\star$ with respect to the reference measure ε_{x_o}, $x_o \ge 0$, that is, the Dirac measure at x_o, is $\bar{I} := [0,+\infty]$.

(ii) For a given finite $\lambda-$excessive function h there exists a unique probability measure $m^h_{x_o}$ on $\bar{I}$ such that for all $x \in I$

$$h(x) = h(x_o)\int_{\bar{I}} K^\star(y;x,x_o)m^h_{x_o}(dy) \tag{6}$$

where the so called *Martin kernel* $K^\star$ is given by

$$K^\star(y;x,x_o) = \begin{cases} \dfrac{g^\star(x,y)}{g^\star(x_o,y)}, & 0 \le y < \infty, \\ \dfrac{\psi^\star(x)}{\psi^\star(x_o)}, & y = \infty. \end{cases}$$

Moreover, the measure $m^h_{x_o}$ is given by

$$m^h_{x_o}((x,\infty]) = \frac{\psi^\star(x_o)}{w^\star\, h(x_o)}\left(\varphi^\star(x)\frac{d^+h}{ds}(x) - h(x)\frac{d\varphi^\star}{ds}(x)\right), \quad x \ge x_o,$$

and

$$m^h_{x_o}([0,x)) = \frac{\varphi^\star(x_o)}{w^\star\, h(x_o)}\left(h(x)\frac{d\psi^\star}{ds}(x) - \psi^\star(x)\frac{d^-h}{ds}(x)\right), \quad x \le x_o,$$

where the differentiation is with respect to the scale function $s(x) = (e^{2\delta x} - 1)/2\delta$ and d^+ (d^-) stand for the right (left) derivative.

The idea is to study these expressions simultaneously for all x_o when h is replaced by r. We have the following preparatory result

Lemma 2. **(i)** The function

$$x \mapsto v(x) := r(x)\frac{d\psi^\star}{ds}(x) - \psi^\star(x)\frac{dr}{ds}(x).$$

is increasing, $\lim_{x\to\infty} v(x) = \infty$, and $v(0) < 0$. The number $a^\star$ given in Theorem 2 is the unique positive solution of the equation $v(x) = 0$.

(ii) The function

$$x \mapsto u(x) := \varphi^\star(x)\frac{dr}{ds}(x) - r(x)\frac{d\varphi^\star}{ds}(x)$$

is non-negative and decreasing.

Proof – being straightforward and elementary – is omitted.

For $x_o \geq a^\star$ introduce

$$m^r_{x_o}((x,\infty]) := \frac{\psi^\star(x_o)}{w^\star\, r(x_o)} u(x), \quad x \geq x_o,$$

and

$$m^r_{x_o}([0,x)) := \begin{cases} \dfrac{\varphi^\star(x_o)}{w^\star\, r(x_o)} v(x), & a^\star \leq x \leq x_o, \\ 0, & 0 \leq x < a^\star. \end{cases}$$

Further, set $m^r_{x_o}(\{x_o\}) = 0$. Then using Lemma 2 and the definition of the Wronskian, see, e.g., Borodin and Salminen (1996) II.11. p. 19, it is seen that $m^r_{x_o}$ is for every $x_o \geq a^\star$ a probability measure on $[0,\infty]$. Notice also that

$$m^r_{x_o}(\{0\}) = m^r_{x_o}(\{a^\star\}) = m^r_{x_o}(\{\infty\}) = 0.$$

Using $m^r_{x_o}$ in the representation formula (6) gives after some straightforward computations a λ–excessive function $x \mapsto h_{x_o}(x)$ such that

$$h_{x_o}(x) = \begin{cases} r(x)/r(x_o), & x \geq a^\star, \\ r(a^\star)\, \psi^\star(x)/\psi^\star(a^\star) r(x_o), & x \leq a^\star. \end{cases}$$

Clearly, the function $x \mapsto h^\star(x) := r(x_o) h_{x_o}(x)$ is independent of x_o, and we prove the following

Proposition 4. The function $h^\star$ is the smallest λ–excessive majorant of r, and, consequently, $V^\star = h^\star$.

Proof. From the construction it is clear that $h^\star$ is λ–excessive. Clearly, $h^\star$ is a majorant of r if $h^\star > r$ on $(0, a^\star)$. To prove this notice that $x \mapsto q(x) := r(x)/\psi^\star(x)$ is increasing on $(0, a^\star)$ because $q'(x) = -v(x)/\psi^\star(x)^2 > 0$, and, hence, for all $x \in (0, a^\star)$

$$q(x) < q(a^\star) \quad \Leftrightarrow \quad r(x)/\psi^\star(x) < r(a^\star)/\psi^\star(a^\star) \quad \Leftrightarrow \quad r(x) < h^\star(x).$$

Assume now that there exists a λ–excessive majorant $\hat{h}$ of r smaller than $h^\star$. Because $h^\star = \hat{h}$ on $[a^\star, \infty)$ it follows from Lemma 2 that the representing measures of $h^\star$ and $\hat{h}$ must be the same. Therefore, $h^\star = \hat{h}$ on $[0, \infty)$ completing the proof. //

Remark. From (6) it is seen (cf. Salminen (1985)) that every λ–excessive function h of $Z^\star$ is continuous and has left and right derivatives which satisfy $h'_-(x) \geq h'_+(x)$ for $x > 0$. Therefore, if h is λ–excessive and $h(x) = r(x)$ for $x \geq a > a^\star$ then there exists $\varepsilon > 0$ such that $h(x) < r(x)$ for $x \in (a - \varepsilon, a)$, and h is not a majorant of r.

To conclude the proof of Theorem 2 we notice that RBM$(-\delta)$ is positively recurrent and, therefore, $\tau_{a^\star} < \infty$ $\mathbf{P}_x^\star$–a.s.

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