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OPTIMAL STOPPING AND AMERICAN PUT OPTIONS

1. INTRODUCTION

In this paper we discuss the pricing of the American put option. We view ourselves as an option seller. The game starts when the stock owner pays us a amount u_o and receives the right (but not the obligation) to sell us her/his stock at the price K at any future time (she/he wishes) however before a given deadline T (which could be infinite). Our task is to create using the initial capital u_o so much wealth that we are able to satisfy the claim, that is, if the option buyer wants to exercise the option at time t our wealth should at that time be at least $(K - S_t)^+$, where S_t is the price of the stock at time t and, as usual, $x^+ := \max\{x, 0\}$. The basic result due to Black, Scholes and Merton is that under some assumptions the option has a fair price in the sense that if the option buyer uses her/his right optimally our wealth is exactly the amount we are asked to pay at the excercision time.

We present next a set of assumptions (in other words, a mathematical model, so called Black-Scholes' market model) under which the option pricing problem has a solution. Our reference is Karatzas (1997).

Let $\mathbf{C}$ be the space of continuous functions $\omega : \mathbf{R}_+ \mapsto \mathbf{R}$ and $\mathcal{C}_t,\ t \geq 0$ the $\sigma-$algebra making the co-ordinate mapping $\{\omega_s\ :\ 0 \leq s \leq t\}$ measurable. Further, let $\mathbf{P}$ denote the probability measure on $(\mathbf{C}, \mathcal{C})$, $\mathcal{C} := \sigma\{\omega_s\ :\ s \geq 0\}$, such that $\{W_t\ :\ t \geq 0\} := \{\omega_t\ :\ t \geq 0\}$ is a standard Brownian motion started at 0. Let $S = \{S_t :\ t \geq 0\}$ be a geometrical Brownian motion started at x with parameters μ and σ, GBM(μ, σ^2). Then S can be realized by defining

$$S := \{S_t := x\ \exp\big((\mu - \sigma^2/2)t + \sigma W_t\big)\ :\quad t \geq 0\}.$$

We view S as the *price process of the stock*, and assume that there exists also a riskless alternative to invest, e.g., on a bond. Let $r > 0$ be the interest rate of the bond. Then the *price process of the bond* when its initial price is y is

$$\{B_t := y\, e^{r\,t}\ :\ t \geq 0\}.$$

As explained above, we consider ourselves as investor having the option's price as our initial capital. It is assumed that our *wealth process*, $\{X_t :\ t \geq 0\}$, obeys the stochastic differential equation

$$dX_t = \pi_t \frac{dS_t}{S_t} + (X_t - \pi_t)\frac{dB_t}{B_t} - dC_t, \tag{1.1}$$

where π_t is the amount invested in stock at time t and $C := \{C_t\ :\ t \geq 0\}$ is the *consumption process.* The *portfolio process* $\pi := \{\pi_t\ :\ t \geq 0\}$ is supposed to be progressively measurable such that for all $t \geq 0$ a.s.

$$\int_0^t |\pi_s| ds < \infty.$$

The consumption process C is assumed to be non-decreasing, right continuous, $\mathcal{C}_t-$adapted and starting at 0. The solution of (1.1) satisfies

$$e^{-rt}X_t - X_0 = (\mu - r)\int_0^t e^{-rs}\pi_s ds + \sigma\int_0^t e^{-rs}\pi_s dW_s - \int_0^t e^{-rs}dC_s.$$

The process given by

$$M_t^\pi := (\mu - r)\int_0^t e^{-rs}\pi_s ds + \sigma \int_0^t e^{-rs}\pi_s dW_s$$

is called the *discounted gains process.* It is natural to restrict the class of possible portfolio processes by demanding that the corresponding gains process is bounded from below. Therefore, a portfolio process π is called *tame* if there exists $q^\pi \in \mathbf{R}$ such that

$$\mathbf{P}(M_t^\pi \geq q^\pi \ \forall\ 0 \leq t \leq T\} = 1,$$

where T is the given time horizon. From now on it is assumed that all considered portfolio processes are tame. A portfolio process π such that

$$\mathbf{P}(M_T^\pi \geq 0) = 1 \ \text{ and } \ \mathbf{P}(M_T^\pi > 0) > 0$$

is called an *arbitrage opportunity.* The first basic result is the following

Theorem 1.1 *In the Black-Scholes' market model there are no arbitrage opportunities.*

The measure $\mathbf{P}^+$ on $(\mathbf{C}, \mathcal{C})$ such that the co-ordinate process $\{\omega_t\ :\ t \geq 0\}$ is a Brownian motion with drift $\dfrac{r-\mu}{\sigma}$ is called the *martingale measure.* This measure is used when we construct so called *hedging strategies.* Notice that under $\mathbf{P}^+$ the discounted gains process M^π is a local martingale (for every portfolio process π). Further, under $\mathbf{P}^+$ the stock price process S is a GBM(r, σ^2) and satisfies the SDE

$$dS_t = rS_t dt + \sigma S_t dW_t^+,$$

where W^+ is a Brownian motion; so, under $\mathbf{P}^+$ the discounted stock price process $\{e^{-rt}S_t\ :\ t \geq 0\}$ is a martingale.

Definition *An European contingent claim is a $\mathcal{C}_T-$measurable random variable $Y\ :\ \mathbf{C} \mapsto \mathbf{R}_+$ such that $u_o := e^{-rT}\mathbf{E}^+(Y) < \infty$.*

Recall that an European contingent claim can be exercised only at the deadline T. We say that an European contingent claim is *attainable* if there exists a portfolio process π such that $X_T = Y$ when the initial wealth $X_0 = u_o$. Such a portfolio process is called a *hedging strategy* for the claim.

Theorem 1.2 *In the Black-Scholes' market model every European contingent claim is attainable.*

As indicated in the title of the paper we focus here on American options.

Definition *An American contingent claim is a $\mathcal{C}_t-$adapted continuous stochastic process $Y\ :\ [0,T]\times\mathbf{C} \mapsto \mathbf{R}_+$ such that*

$$\mathbf{E}^+(\sup_{0\leq t\leq T} e^{-rt}Y_t) < \infty.$$

Also all American contingent claims are attainable in the sense given in the next

Theorem 1.3 *Let Y be an American contingent claim in the Black-Scholes market model. Then*

$$\begin{aligned} u_o^\star :&= \sup\{\mathbf{E}^+(e^{-r\tau}Y_\tau)\ :\ \tau \in \mathcal{M}_{0,T}\} \\ &= \inf\{u\ :\ \exists \pi, C \text{ such that } X_t \geq Y_t\ \forall 0 \leq t \leq T \text{ a.s. and } X_0 = u\}, \end{aligned}$$

where $\mathcal{M}_{0,T}$ is the set of stopping times τ such that $0 \leq \tau \leq T$. Moreover, there exist a portfolio process $\pi^\star$ and a consumption process $C^\star$ such that the corresponding wealth process $X^\star$ satisfies

$$\begin{aligned} &X_t^\star \geq Y_t \quad \text{for all } 0 \leq t \leq T \\ &X_T^\star = Y_T \\ &X_0^\star = u_o^\star \end{aligned}$$

and can be defined as

$$X_t^\star := \operatorname{ess\,sup}\{\mathbf{E}^+(e^{-r(\tau-t)}Y_\tau|\mathcal{C}_t) \,:\, \tau \in \mathcal{M}_{t,T}\}.$$

The stopping time

$$\tau^\star := \inf\{t \le T \,:\, X_t^\star = Y_t\}$$

realizes the supremum $u_o^\star$.

In this paper we analyze in details American contingent claim with $Y_t = (K - S_t)^+$, that is, the classical American put option. However, because we focus on optimal stopping, hedging strategies are not constructed, for these see Karatzas (1997). In Section 3 the infinite horizon case $T = \infty$ is discussed and in Section 4 we treat the finite horizon case $T < \infty$.

Of course, the American put option has been very much studied, and the results we present here can be found also in some textbooks, see, e.g., Karatzas and Shreve (1998) (also for further references). The new feature we are offering is a technique based on the representation theory of excessive functions when doing optimal stopping. Especially, in the infinite horizon case it is seen that our method gives the solution more directly than the much used techniques based on the principle of smooth pasting (also called the approach of variational inequalities). In fact, the smooth pasting condition has a simple explanation via properties of excessive functions (see Remark 3.1). We hope to convince our readers that the representation theory is a valuable tool when solving optimal stopping problems. In the next section we give a short introduction to the representation theory of excessive functions, and apply these results in Sections 3 and 4 for optimal stopping. The method was developped in Salminen (1985) but has not received much interest.

In the finite horizon case the explicit analytical solution of the optimal stopping problem is not known, that is, the stopping boundary does not seem to have a "simple" analytical form although many properties of the boundary are known. In Jacka (1991) an integral equation is derived for which the function determining the stopping boundary is the unique solution. The representation theory of excessive functions can be used to give a new derivation of this equation. The decomposition of the value of the American put option as the sum of the European put option and an early exercise premium is seen to be nothing but the representation formula for the value function. It is also seen that the expression for the consumption process C of the option seller is easily optained from this formula.

Finally we want to stress that the presented method can also be applied similarly for other options like, e.g., barrier put options $Y_t = (K - S_t)^+ \mathbf{1}_{\{t<H_a\}}$, $H_a := \inf\{t \,:\, S_t = a\}$ and Russian options $Y_t = e^{-\lambda t}\sup_{s\le t} S_s$. These as well as some technical details of the method are treated in a forthcoming paper.

2. OPTIMAL STOPPING AND EXCESSIVE FUNCTIONS

We consider only geometric Brownian motion although the definitions and properties presented in this section are clearly very general. Let, as in Introduction, $S := \{S_t \,:\, t \ge 0\}$ be GBM(r, σ^2), and define also space-time GBM on the time interval $(0, T)$ via $\bar{S} := \{(t, S_t) \,:\, 0 \le t < T\}$. Recall that S can be viewed as the stock price process under the martingale measure $\mathbf{P}^+$. For the measure associated with $\bar{S}$ when started at (s, x) we use the notation $\mathbf{P}^+_{(s,x)}$.

Definition

(a) A Borel-measurable function $h : (0,T) \times \mathbf{R}_+ \mapsto \mathbf{R}_+$ *is called* $\lambda-$*excessive* $(\lambda \ge 0)$ *for* $\bar{S}$ *if for all* $x > 0$ *and* $0 \le s < T$

$$\lim_{t\downarrow s} \mathbf{E}^+_{(s,x)}(e^{-\lambda(t-s)}h(t, S_t)) \uparrow h(s,x).$$

(b) A Borel-measurable function $h : (0, +\infty) \mapsto \mathbf{R}_+$ *is called* $\lambda-$*excessive for* S *if for all* $x > 0$

$$\lim_{t\downarrow 0} \mathbf{E}^+_x(e^{-\lambda t}h(S_t)) \uparrow h(x).$$

The following result is from Shiryayev (1978) (Theorem 1 p. 124 and Theorem 3 p. 127). We formulate it for $\bar{S}$.

Theorem 2.1 *Assume that $(t,x) \mapsto f(s,x)$ is continuous, bounded and non-negative. Then the smallest $\lambda-$excessive majorant of f is given by*

$$v(s,x) := \sup \mathbf{E}^{+}_{(s,x)}(e^{-\lambda(\tau-s)} f(\tau, S_\tau)),$$

where the supremum is taken with respect to stopping times $\tau \in \mathcal{M}_{s,T}$, i.e., $s \leq \tau \leq T$. The supremum is realized by the stopping time

$$\tau^\star := \inf\{t \,:\, (t, S_t) \in \Gamma\},$$

where $\Gamma := \{(t,x) \,:\, v(t,x) = f(t,x)\}$ is the so called stopping region or exercise region (the set $\Delta := \{(t,x) \,:\, v(t,x) > f(t,x)\}$ is called the continuation region).

Proposition 2.1 *The smallest $\lambda-$excessive majorant v is smooth on Δ and satisfies*

$$\mathcal{A}_\lambda v(t,x) := \frac{\partial v}{\partial t}(t,x) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 v}{\partial x^2}(t,x) + rx \frac{\partial v}{\partial x}(t,x) - \lambda v(t,x) = 0. \tag{2.1}$$

In other words, v is $\lambda-$harmonic on Δ, i.e.,

$$v(s,x) = \mathbf{E}^{+}_{(s,x)}(e^{-\lambda\tau_A} v(\tau_A, S_{\tau_A})),$$

where A is an open subset of Δ and $\tau_A := \inf\{t \,:\, (t, S_t) \notin A\}$.

Proof. From Shiryayev (1978) Corollary 1 p. 128 we have

$$v(s,x) = \mathbf{E}^{+}_{(s,x)}(e^{-\lambda\tau^\star} v(\tau^\star, S_{\tau^\star})) = \mathbf{E}^{+}_{(s,x)}(e^{-\lambda\tau^\star} f(\tau^\star, S_{\tau^\star})),$$

and the claim follows from the strong Markov property.

Remark. *On the stopping region v is typically $\lambda-$superharmonic which implies (under smoothness assumption) that v satisfies on Γ*

$$\mathcal{A}_\lambda v(t,x) \leq 0. \tag{2.2}$$

Now (2.1) and (2.2) together with

$$v > f \text{ on } \Delta$$

$$v = f \text{ on } \Gamma$$

consitute essentially the so called variational inequalities of the optimal stopping problem the objective being to characterize the sets Δ and Γ by determining their boundaries. For this approach see Oeksendal (1998) or Karatzas and Shreve (1998).

We proceed by presenting the representation theorems for S and $\bar{S}$. Consider first the λ-excessive functions of S. To state the result we need some notation. Let

$$c_\pm := \frac{1-\gamma}{2} \pm \sqrt{\left(\frac{1-\gamma}{2}\right)^2 + \frac{2\lambda}{\sigma^2}}, \quad \gamma := \frac{2r}{\sigma^2},$$

and define for $x > 0$

$$\varphi_\lambda(x) := x^{c_-}, \quad \psi_\lambda(x) := x^{c_+}.$$

Let also $w_\lambda := c_+ - c_-$. Then the Green function of S can be expressed (see Borodin and Salminen (1996) p. 110) as

$$g_\lambda(x,y) := \int_0^\infty e^{-\lambda t} p(t;x,y)dt = \frac{\psi_\lambda(x)\varphi_\lambda(y)}{w_\lambda}, \quad 0 \leq x \leq y,$$

where p is the symmetric transition density of S (with respect to the speed measure). The following is our first representation theorem

Theorem 2.2 *(i) The Martin compactification of the state space* $I = (0, \infty)$ *of* S *with respect to the reference measure* ε_{x_o}, $x_o > 0$, *that is, the Dirac measure at* x_o, *is* $\bar{I} := [0, +\infty]$.

(ii) For a given finite $\lambda-$*excessive function* h *there exists a unique probability measure* $m^h_{x_o}$ *on* $\bar{I}$ *such that for all* $x \in I$

$$h(x) = h(x_o) \int_{\bar{I}} K_\lambda(y; x, x_o) m^h_{x_o}(dy) \tag{2.3}$$

where

$$K_\lambda(y; x, x_o) = \begin{cases} \dfrac{\varphi_\lambda(x)}{\varphi_\lambda(x_o)}, & y = 0, \\ \dfrac{g_\lambda(x, y)}{g_\lambda(x_o, y)}, & 0 < y < \infty, \\ \dfrac{\psi_\lambda(x)}{\psi_\lambda(x_o)}, & y = \infty, \end{cases}$$

is the so called Martin kernel. Moreover, the measure $m^h_{x_o}$ *is given by*

$$m^h_{x_o}((x, \infty]) = \frac{\psi_\lambda(x_o)}{w_\lambda\, h(x_o) s'(x)} \big(\varphi_\lambda(x) \frac{d^+ h}{dx}(x) - h(x) \frac{d\varphi_\lambda}{dx}(x)\big), \quad x \geq x_o,$$

and

$$m^h_{x_o}([0, x)) = \frac{\varphi_\lambda(x_o)}{w_\lambda\, h(x_o) s'(x)} \big(h(x) \frac{d\psi_\lambda}{dx}(x) - \psi_\lambda \frac{d^- h}{dx}(x)\big), \quad x \leq x_o,$$

where $s'(x) = x^{-\gamma}$ *is the derivative of the scale function and* d^+ (d^-) *stands for the right (left) derivative.*

Proof. Dynkin (1969) and Kunita and Watanabe (1965) treat the Martin boundary theory for very general Markov processes. The result here can be obtained, e.g., from Theorem 3 p. 509 and Theorem 4 p. 513 in Kunita and Watanabe (1965). In Dynkin (1969) Teorema 8.2 a formula (displayed below in (2.5)) for computing the representing measure is given. The above presented form of the representing measure can be found in Salminen (1985). For a discussion of Martin boundary theory especially for linear diffusions see Salminen (1984).

Corollary 2.1 *Every* $\lambda-$*excessive function of* S *is continuous and has left and right derivatives which satisfy* $h'_-(x) \geq h'_+(x)$. *In particular,* h *is differentiable at* x *if and only if the representing measure of* h *does not charge* x.

Proof. See Salminen (1985).

Next we give the representation theorem for the $\lambda-$excessive functions of $\bar{S}$.

Theorem 2.3 *(i) The Martin compactification of the state space* $I = (0, T) \times (0, \infty)$ *of* $\bar{S}$ *with respect to the reference measure* $\varepsilon_{(0, x_o)}$, $x_o > 0$ *is* $\bar{I} := (0, T] \times (0, \infty)$.

(ii) For a given finite $\lambda-$*excessive function* h *there exists a unique probability measure* $m^h_{x_o}$ *on* $\bar{I}$ *such that*

$$h(s, x) = h(0, x_o) \int_{(0,T]} \int_{(0, +\infty)} m^h_{x_o}(dt, dy) K(t, y; s, x, x_o), \tag{2.4}$$

where

$$K(t, y; s, x, x_o) = \begin{cases} \dfrac{e^{-\lambda(t-s)} p(t - s; x, y)}{e^{-\lambda t} p(t; x_o, y)}, & s < t \leq T, \\ 0, & t \leq s < T, \ x \neq y, \\ +\infty, & t = s < T, \ x = y. \end{cases}$$

Proof. The statement follows from the general Martin boundary theory (see Dynkin (1969) and Kunita and Watanabe (1965)). The explicit Martin compactification can be computed using properties of the transition function (see Salminen (1981) for some cases). Notice that $\{(t,0) : 0 < t \le T\}$ is not a part of the compactification.

Proposition 2.2 *Let h be a bounded $\lambda-$excessive function for $\bar{S}$ such that $h \in C^b_{1,2}(F)$, i.e., $\partial h/\partial t$, $\partial h/\partial x$ and $\partial^2 h/\partial x^2$ are continuous and bounded inside the compact subset F of $(0,T) \times \mathbf{R}_+$. Then the representing measure of h is inside F absolutely continuous with respect to the Lebesque measure and its derivative $m^h_{x_o}$ is given by*

$$m^h_{x_o}(s,x) = -e^{-\lambda s} p(s; x_o, x) \mathcal{A}_\lambda h(s,x),$$

where (s,x) is an inner point of F and $\mathcal{A}_\lambda$ is as given in (2.1).

Proof. Recall that (see Dynkin (1969), Teorema 8.2) if $g : (s,x) \mapsto \mathbf{R}_+$ is continuous with compact support inside F we have

$$\int_{(0,T)\times\mathbf{R}_+} g(s,x)\, m^h_{x_o}(ds,dx) = \lim_{t\to 0} \int_0^T ds \int_0^\infty dx\, g(s,x) e^{-\lambda s} p(s; x_o, x) \frac{h(s,x) - \mathbf{E}^+_{(s,x)}(e^{-\lambda t} h(s+t, S_{s+t}))}{t}. \tag{2.5}$$

Because $h \in C^b_{1,2}(F)$ we obtain for an inner point (s,x) of F using, e.g., a stopping argument and Ito's formula

$$\lim_{t\to 0} \frac{h(s,x) - \mathbf{E}_{(s,x)}(e^{-\lambda t} h(s+t, S_{s+t}))}{t} = -\mathcal{A}_\lambda h(s,x).$$

Consequently, we can take the limit inside the integral sign proving the claim.

3. AMERICAN PUT OPTION WITH INFINITE HORIZON

The task is to find

$$v(x) := \sup_\tau \mathbf{E}^+_x(e^{-r\tau}(K - S_\tau)^+)$$

and characterize the stopping time realizing the supremum. Recall that S is GBM(r,σ^2) (under the martingale measure $\mathbf{P}^+$). Let $f(x) := (K-x)^+$ and introduce the function (cf. Theorem 2.2; note that $\lambda = r$ and so $c_+ = 1$ and $c_- = -\gamma$)

$$u(x) := \frac{1}{s'(x)}\Big(\varphi_r(x)\frac{d^+ f}{dx}(x) - f(x)\frac{d\varphi_r}{dx}(x)\Big)$$

$$= \begin{cases} 0, & x > K, \\ x^{-1}(K\gamma - (1+\gamma)x), & x < K \end{cases}$$

It is obvious that $x \mapsto u(x)$ is decreasing when $x < K$, $u(x) \to +\infty$ when $x \to 0$, and $u(K-) < 0$. Consequently, $x^\star := \dfrac{K\gamma}{1+\gamma}$ is the unique solution of the equation $u(x) = 0$ on $(0,K)$. Next consider the function

$$t(x) := \frac{1}{s'(x)}\Big(f(x)\frac{d\psi_r}{dx}(x) - \psi_r\frac{d^- f}{dx}(x)\Big)$$

$$= \begin{cases} 0, & x > K, \\ K\, x^\gamma, & x < K. \end{cases}$$

Then $t(0) = 0$ and t is increasing on $(0,K)$. Choose now the reference point $x_o < x^\star$ and define

$$m^f_{x_o}((x,+\infty]) := \begin{cases} 0, & x \ge x^\star, \\ \dfrac{\psi_r(x_o)}{w_r\, f(x_o)} u(x), & x_o \le x \le x^\star, \end{cases}$$

$$m^f_{x_o}([0,x)) := \frac{\varphi_r(x_o)}{w_r\, f(x_o)} t(x), \quad x \leq x_o.$$

By the properties of u and t these definitions are sufficient to give us a Borel–measure on $\mathbf{R}_+$. Using the definition of Wronskian w_r (see e.g. Borodin and Salminen (1996) II.11. p. 19) we obtain

$$m^f_{x_o}([0,x_o)) + m^f_{x_o}((x_o,+\infty]) = 1.$$

Therefore, setting $m^f_{x_o}(\{x_o\}) = 0$ makes $m^f_{x_o}$ to a probability measure. Notice also that

$$m^f_{x_o}(\{0\}) = \lim_{x\to 0} m^f_{x_o}([0,x)) = 0,$$

$$m^f_{x_o}(\{+\infty\}) = \lim_{x\to+\infty} m^f_{x_o}((x,+\infty]) = 0.$$

The probability measure $m^f_{x_o}$ induces via the representation formula (2.3) the $r-$excessive function

$$h_{x_o}(x) = \begin{cases} f(x)/f(x_o), & x \leq x^\star, \\ f(x^\star)\,\varphi_r(x)/f(x_o)\varphi_r(x^\star), & x \geq x^\star. \end{cases}$$

Clearly, the function $x \mapsto h^\star(x) := f(x_o)h_{x_o}(x)$ is independent of x_o, and we have

Proposition 3.1 *The function $h^\star$ is the smallest r–excessive majorant of f.*

Proof. From the construction it is clear that $h^\star$ is r–excessive. Clearly, $h^\star$ is a majorant of f if $h^\star > f$ on $(x^\star,+\infty)$. To prove this consider for $x \in (x^\star, K)$ the function $q(x) := f(x)/\varphi_r(x)$. Then, because $q'(x) = s'(x)u(x)/\varphi_r^2(x) < 0$ on $(x^\star, K)$, q is decreasing on $(x^\star, K)$, and so on $(x^\star, K)$ we have

$$\frac{f(x^\star)}{\varphi_r(x^\star)} > \frac{f(x)}{\varphi_r(x)} \quad \Leftrightarrow \quad h^\star(x) > f(x).$$

Assume now that there exists an r–excessive majorant $\hat{h}$ of f smaller than $h^\star$. Because $h^\star = \hat{h}$ on $(0,x^\star)$ it follows from the properties of u and t that the representing measures of $h^\star$ and $\hat{h}$ must be the same. Therefore, $h^\star = \hat{h}$ on $(0,\infty)$ completing the proof.

Remark 3.1 *In fact, $h^\star$ is the only r–excessive majorant of f which is equal to f on an interval. This follows from Corollary 2.1. Indeed, the interval, where f and its majorant h coincide, is by the properties of u and v a part of $(0,x^\star)$. Hence, the representing measure of h puts mass at the endpoint(s) of the interval, and therefore h is not differentiable at the endpoint(s). Because at the endpoint $h'_- > h'_+$ it follows that h is not a majorant of f. Recall also that the principle of smooth pasting says that the smallest $r-$excessive majorant is differentiable at the boundary points of the continuation region. Our approach explains, therefore, why this principle holds.*

4. AMERICAN PUT OPTION WITH FINITE HORIZON

Here we consider the optimal stopping problem

$$\sup_{\tau\in\mathcal{M}_{0,T}} \mathbf{E}^+_x(e^{-r\tau}(K-S_\tau)^+) \tag{4.1}$$

where $\mathcal{M}_{s,T}$ is for $0 \leq s < T$ the set of stopping times with values in $(s,T]$. Let

$$v(s,x) := \sup_{\tau\in\mathcal{M}_{s,T}} \mathbf{E}^+_{s,x}(e^{-r\tau}(K-S_\tau)^+).$$

be the value function of the problem. It does not seem to be possible to find an explicit analytical solution to the problem (but there are, of course, a variety of numerical methods). In the next theorem we have collected results concerning the problem. Proofs can be found in Jacka (1991) (see also Karatzas and Shreve (1998), Karatzas (1996) and Myneni (1992)).

Theorem 4.1 *Let $\Delta := \{(s,x) : v(s,x) > (K-x)^+\}$ and $\Gamma := \{(s,x) : v(s,x) = (K-x)^+\}$ be the continuation and exercise region, respectively, of the stopping problem (4.1). Then there exists a continuous and increasing function $s \mapsto b(s)$ such that*

a) $b(0) < K, \quad b(T) = K, \quad \lim_{T\to\infty} b(0) = \dfrac{-Kc_-}{1-c_-} =: x^\star.$

b) $\Delta = \{(s,x) : x > b(s)\}$

c) $\Gamma = \{(s,x) : x \le b(s)\}$

d) The price of the option at time s when $S_s = x$ has the decomposition

$$v(s,x) = rK \int_s^T e^{-r(t-s)} \mathbf{P}^+_{s,x}(S_t < b(t))dt + \mathbf{E}^+_{s,x}(e^{-r(T-s)}(K-S_T)^+).$$

e) The consumtion process of the option seller is

$$C_t = rK \int_0^t \mathbf{1}_{\{v\,:\,S_v < b(v)\}}(u)du, \quad t \ge 0.$$

f) for any $s > 0$ the function $b(t)$, $t \ge s$, is the unique (left) continuous solution of the integral equation

$$K - x = rK \int_s^T e^{-r(t-s)} \mathbf{P}_{(s,x)}(S_t < b(t))dt + \mathbf{E}_{(s,x)}(e^{-r(T-s)}(K - S_{T-s})^+), \tag{4.2}$$

where $x \le b(s)$.

Remark. *Notice that the second term on the right hand side of the equation in d) is the value of the European put option; hence, the first term is called the early exercise premium for the American put option.*

Our aim is to study the results in Theorem 4.1 from the point of view of the representation theory. Therefore, *we assume that a), b) and c) hold,* (intuitively, these are very understandable). Under this assumption, as shown below, the statement d) is simply the representation formula. From d) we get Jacka's equation f). The representation theory does not, however, seem to give uniqueness. To make paper more self contained we reproduce Jacka's proof (with the motivation that the equation in f) looks different than the equation in Jacka (1991) because the "direction in $t-$variable" is reversed). The proof of f) is based on the properties of some martingales appearing in our derivation of the consumption process. Hence, we prove e) before f). Another proof of e), based on fairly delicate stochastic calculus, can be found in Karatzas and Shreve (1998).

Proof of d). To start with we normalize v by introducing the function $h(s,x) := v(s,x)/v(0,x_o)$, which clearly takes the value 1 at the reference point $(0,x_o)$. Now we consider the representation of h.

i) Recall from Proposition 2.1 that on Δ the value function v, and hence also h, is $r-$harmonic. Therefore, because $\mathcal{A}_r h = 0$ on Δ it follows that the the representing measure of h does not charge Δ.

ii) h is smooth on every open subset of Γ (being there equal to $f(s,x) := K - x$). Hence, using again Proposition 2.2, we compute therein its representing measure

$$\begin{aligned} m^h_{x_o}(s,x) &= \frac{-e^{-rs}p(s;x_o,x)}{v(0,x_o)}\mathcal{A}_r f(s,x) \\ &= \frac{e^{-rs}p(s;x_o,x)}{v(0,x_o)} rK \end{aligned}.$$

iii) From the definition of v it follows

$$h(T-,x) = \frac{(K-x)^+}{v(0,x_o)}.$$

Consequently, the representation formula gives us

$$m^h_{x_o}(\{T\},dy) = e^{-rT}p(T;x_o,y)\frac{(K-y)^+}{v(0,x_o)}dy.$$

iv) To complete the description it remains to study the boundary of Γ, i.e. the set $\Gamma_b := \{(s,x) : x = b(s), 0 \le s \le T\}$. The claim is that $m^h_{x_o}(\Gamma_b) = 0$, that is, the smooth pasting condition holds, cf. Remark 3.1. First notice that m^h cannot charge a point on Γ_b because then the value function would attain the "value" $+\infty$ at this point which is impossible. Assume next that for some $s_1 < s_2 < T$ the set $L := \{(s,x) : x = b(s), u < s < v\}$ has a positive m^h-mass for all u and v such that $s_1 < u < v < s_2$. Then it can be proved that $\partial h/\partial x$ is discontinuous at the points $(s,b(s)) \in L$. In fact, we have

$$\frac{\partial h}{\partial x}(s,b(s)-) > \frac{\partial h}{\partial x}(s,b(s)+). \tag{4.3}$$

But (4.3) implies that v attains in a neigbourhood of L values smaller than the reward function $f(s,x) = (K-x)^+$ violating the fact that v is a majorant of f. A proof of (4.3) is given in a forthcoming paper (where also other diffusions are discussed). It is quite obvious that (4.3) is inherited from the characteristic property of Green kernels; see the example after the proof.

Combining i)-iv) gives us the following representation

$$\begin{aligned} v(s,x) &= rK\int_s^T dt\int_0^{b(t)} dy\, e^{-r(t-s)}p(t-s;x,y) + \int_0^{b(T)} dy\, e^{-r(T-s)}p(T-s;x,y) \\ &= rK\int_s^T e^{-r(t-s)}\mathbf{P}^+_{s,x}(S_t < b(t))\,dt + \mathbf{E}^+_{s,x}(e^{-r(T-s)}(K-S_T)^+). \end{aligned}$$

as claimed in d).

Illuminating example. To convince the reader about (4.3) we give a simple example with standard Brownian motion. So, let p be the transition density of the standard Brownian motion and m a measure on the set $\{(s,0) : s_1 < s < s_2\}$ Suppose that m is absolutely continuous with respect to the Lebesque measure. Let m denote also the derivative of m and assume that m is bounded and continuous. Define

$$h(s,x) := \int_s^{s_2} m(t)e^{rs}\frac{p(t-s;x,0)}{p(t;x_o,0)}dt.$$

Then for $x \neq 0$ we can differentiate under the integral sign to obtain

$$\frac{\partial}{\partial x}h(s,x) = e^{rs}\int_s^{s_2}\mu(t)\frac{\partial}{\partial x}p(t-s;x,0)dt$$

with $\mu(t) := m(t)/p(t;x_o,0)$. Because

$$\int_s^{s_2}\frac{\partial}{\partial x}p(t-s;x,0)dt = \begin{cases} -\mathbf{P}^{BM}_x(H_0 < s_2 - s), & x > 0, \\ \mathbf{P}^{BM}_{-x}(H_0 < s_2 - s), & x < 0, \end{cases}$$

it is seen (in this case) that

$$\frac{\partial}{\partial x}h(s,0-) > 0 > \frac{\partial}{\partial x}h(s,0+).$$

Proof of e). Recall that the investor's (or option seller's) wealth process, is given by

$$e^{-rt}X_t - X_0 = (\mu - r)\int_0^t e^{-rs}\pi_s ds + \sigma\int_0^t e^{-rs}\pi_s dW_s - \int_0^t e^{-rs}dC_s.$$

Under the martingale measure the discounted gains process

$$M_t^\pi := (\mu - r)\int_0^t e^{-rs}\pi_s ds + \sigma\int_0^t e^{-rs}\pi_s dW_s$$

is a local martingale. On the other hand, we have $X_t = v(t, S_t), 0 \le t \le T$. Therefore, we can find the consumption process by exploiting the Doob-Meyer decomposition of the supermartingale $\{e^{-rt}v(t, S_t) : 0 \le t \le T\}$ (under the martingale measure). To do this we use d) to obtain

$$\begin{aligned} e^{-rt}v(t, S_t) - v(0, x) = {} & rK\int_t^T e^{-ru}\mathbf{P}^+_{(t,S_t)}(S_u < b(u))du + \mathbf{E}^+_{(t,S_t)}(e^{-rT}(K - S_T)^+) \\ & - rK\int_0^T e^{-ru}\mathbf{P}^+_{(0,x)}(S_u < b(u))du - \mathbf{E}^+_{(0,x)}(e^{-rT}(K - S_T)^+). \end{aligned}$$

From the Markov property it follows that

$$M_t^{(1)} := \mathbf{E}^+_{(t,S_t)}(e^{-rT}(K - S_T)^+) - \mathbf{E}^+_{(0,x)}(e^{-rT}(K - S_T)^+),\ 0 \le t \le T,$$

is a martingale. Let next

$$\Lambda_t := rK\int_0^t e^{-ru}\mathbf{1}_{\{v\,:\,S_v < b(v)\}}(u)du,$$

and

$$V_t := rK\int_t^T e^{-ru}\mathbf{P}^+_{(t,S_t)}(S_u < b(u))du - rK\int_0^T e^{-ru}\mathbf{P}^+_{(0,x)}(S_u < b(u))du.$$

Then $M_t^{(2)} := V_t + \Lambda_t$, $0 \le t \le T$, is a martingale. Indeed, we have

$$\begin{aligned} \mathbf{E}(V_t - V_s|\mathcal{F}_s) &= -rK\int_s^t e^{-ru}\mathbf{P}_{(s,S_s)}(S_u < b(u))du \\ &= -\mathbf{E}(\Lambda_t - \Lambda_s|\mathcal{F}_s), \end{aligned}$$

because

$$\Lambda_t - \Lambda_s = rK\int_s^t e^{-ru}\mathbf{1}_{\{v\,:\,S_v < b(v)\}}(u)du.$$

Consequently, we have

$$e^{-rt}X_t - X_0 = M_t^{(1)} + M_t^{(2)} - \Lambda_t,$$

and by the uniqueness of the Doob-Meyer decomposition we argue that

$$C_t = rK\int_0^t \mathbf{1}_{\{v\,:\,S_v < b(v)\}}(u)du.$$

Proof of f). Because $v(s, x) = K - x$ when $x \le b(s)$ it follows from d) that b solves the stated equation. We proceed by presenting a modification of Jacka's proof of uniqueness (see Jacka (1991)) which fits to our point of view. Let now $s > 0$ be given. Then as shown in e)

$$M_t := e^{-rt}v(t, S_t) + rK\int_s^t e^{-ru}\mathbf{1}_{\{v\,:\,S_v < b(v)\}}(u)du,\ t \ge s,$$

is a martingale with respect to $\mathbf{P}^+_{(s,x)}$. Assume now that (4.2) has another solution $b^{(1)}$ such that $b^{(1)}(t) \le K$ for all $t \in (0,T]$, and define

$$v^{(1)}(t,x) := rK \int_t^T e^{-r(u-t)} \mathbf{P}_{(t,x)}(S_u < b^{(1)}(u))du + \mathbf{E}_{(t,x)}(e^{-r(T-t)}(K - S_{T-t})^+).$$

Then also

$$M_t^{(1)} := e^{-rt} v^{(1)}(t, S_t) + rK \int_s^t e^{-ru} \mathbf{1}_{\{v\,:\,S_v < b^{(1)}(v)\}}(u)du$$

is a martingale with respect to $\mathbf{P}^+_{(s,x)}$. Let

$$\tau_1 := \inf\{t \,:\, S_t \le b^{(1)}(t)\} \wedge T$$

Then, for $x > b^{(1)}(s)$,

$$e^{-rs} v^{(1)}(s,x) = M_s^{(1)} = \mathbf{E}_{(s,x)}(e^{-r\tau_1} v^{(1)}(\tau_1, S_{\tau_1}) = \mathbf{E}_{(s,x)}(e^{-r\tau_1}(K - S_{\tau_1})) \le e^{-rs} v(s,x), \qquad (4.4)$$

by the definition of $v^{(1)}$ and the fact that v solves the optimal stopping problem. For $x \le b^{(1)}(s)$ we obtain using again (4.2) and the fact that v is the majorant of $(K-x)^+$

$$v^{(1)}(s,x) = K - x \le v(s,x) \qquad (4.5)$$

On the other hand, taking $x \le b^{(1)}(s) \wedge b(s)$ and setting

$$\tau := \inf\{t \,:\, S_t \ge b(t)\} \wedge T$$

we have by (4.2) and optional sampling

$$\begin{aligned} 0 &= e^{-rs}(v(s,x) - v^{(1)}(s,x)) \\ &= \mathbf{E}_{(s,x)}(e^{-r\tau}(v(\tau, S_\tau) - v^{(1)}(\tau_1, S_{\tau_1}))) + rK\mathbf{E}_{(s,x)}(\int_s^\tau e^{-ru} \mathbf{1}_{\{v\,:\,S_v > b^{(1)}(v)\}}(u))du. \end{aligned}$$

Obviously the second term on the right hand side is non-negative but so is also the first one by (4.4) and (4.5). It follows that $b^{(1)}(t) \ge b(t)$ for $t \in (s,T]$, and so, from the definition of $v^{(1)}$ and the representation of v, we must have $v^{(1)}(s,x) \ge v(s,x)$. Therefore, $v^{(1)}(s,x) = v(s,x)$, which shows that $b^{(1)} = b$ (by left continuity).

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