On occupation times of stationary excursions

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August 11, 2004

Abstract

In this paper excursions of a stationary diffusion in stationary state are studied. In particular, we compute the joint distribution of the occupation times $I_t^{(+)}$ and $I_t^{(-)}$ above and below, respectively, the observed level at time t during an excursion. We consider also the starting time g_t and the ending time d_t of the excursion (straddling t) and discuss their relations to the Lévy measure of the inverse local time. It is seen that the pairs $(I_t^{(+)}, I_t^{(-)})$ and $(t - g_t, d_t - t)$ are identically distributed. Moreover, conditionally on $I_t^{(+)} + I_t^{(-)} = v$, the variables $I_t^{(+)}$ and $I_t^{(-)}$ are uniformly distributed on $(0, v)$. Using the theory of the Palm measures, we derive an analoguous result for excursion bridges.

MSC: 60J60; 60G10
Keywords: Diffusion; Excursion bridge; Feynman-Kac formula; Local time; Palm measure; Ray-Knight theorem; Reflecting Brownian motion with drift

1 Introduction

In the literature there are many examples of cases such that the occupation time for one diffusion is identical in law with the first hitting time for another diffusion. For these see, e.g., [7, 3, 12, 20, 34, 9, 10, 25, 31]. The last one of these references contains a survey of known identities.

In this paper we prove a new kind of identity between hitting and occupation times for stationary diffusions in stationary state. An example of such a diffusion is reflecting Brownian motion on $\mathbb{R}_+$ with drift $-\mu < 0$. As is well known, this process is stationary having the exponential distribution with parameter 2μ as its stationary probability distribution. We take the whole of $\mathbb{R}$ to be the time axis and use, for a moment, the notation $\{X_t : t \in \mathbb{R}\}$ for this process. For fixed $t \in \mathbb{R}$ let

$$g_t := \sup\{s \le t : X_s = 0\}, \quad d_t := \inf\{s > t : X_s = 0\}, \tag{1.1}$$

and

$$I_t^{(+)} := \int_{g_t}^{d_t} \mathbf{1}_{\{X_s > X_t\}}\, ds, \quad I_t^{(-)} := \int_{g_t}^{d_t} \mathbf{1}_{\{0 < X_s \le X_t\}}\, ds.$$

Then from [29] it can be deduced that

$$(I_t^{(+)}, I_t^{(-)}) \stackrel{d}{=} (t - g_t, d_t - t). \tag{1.2}$$

For an explicit form of this distribution and its Laplace transform, see Section 5, Example 5.1. The main result of the present paper states that (1.2) is valid for *all* stationary diffusions in stationary state. By time reversal, the variables $I_t^{(+)}, I_t^{(-)}, t - g_t$, and $d_t - t$ are identically distributed. Our proof of (1.2) is purely computational and does not, unfortunately, provide any probabilistic explanation of the identity. In the case of the reflecting Brownian motion with drift we have an alternative, more probabilistic, proof based on the Ray–Knight theorems. It is fairly easy to deduce also in the general case using the symmetry and time reversal as is done in [29], that the expectations of $I_t^{(+)}, I_t^{(-)}, t - g_t$, and $d_t - t$ are equal.

It is of interest to recall the identity due to [3] and [12] because the variable $d_t - t$ can also be found in this identity. Let $\{B_t^{(\mu)} : t \ge 0\}$ be a Brownian motion with drift $\mu > 0$ starting from 0 and

$$H_x(B^{(\mu)}) := \inf\{t : B_t^{(\mu)} = x\}$$

the first hitting time of the level $x > 0$. Then the Biane–Imhof identity states that

$$\int_0^\infty \mathbf{1}_{\{B_s^{(\mu)} < 0\}}\, ds \quad \stackrel{(d)}{=} \quad H_\lambda(B^{(\mu)}), \tag{1.3}$$

where λ is exponentially (with parameter 2μ) distributed random variable independent of $B^{(\mu)}$. By spatial homogeneity, we have

$$H_\lambda(B^{(\mu)}) \stackrel{\text{(d)}}{=} d_t - t$$

since the distribution of X_t is exponential with parameter 2μ.

From the identity (1.2) using the theory of Palm measures applied for excursions we derive a new interesting result for excursion bridges. To explain this, let $X^{(0,l,0)}$ be an excursion bridge of length l for excursions from 0 to 0 for an arbitrary diffusion X. Further, let U be a random variable having the uniform distribution on $(0,l)$ and assume that U is independent of $X^{(0,l,0)}$. Then the occupation times

$$I^{(l,+)} := \int_0^l \mathbf{1}_{\{X_s^{(0,l,0)} > X_U^{(0,l,0)}\}}\, ds, \quad I^{(l,-)} := \int_0^l \mathbf{1}_{\{X_s^{(0,l,0)} < X_U^{(0,l,0)}\}}\, ds$$

are identically uniformly distributed on $(0,l)$. For a standard Brownian excursion, i.e., for the 3-dimensional Bessel bridge, this fact can also be explained via Vervaat's path transformation.

The paper is organized so that there are two main sections and two shorter section with remarks and examples. The first main section is devoted to the identity (1.2) and is divided into four subsections. In the first subsection some preliminaries on stationary diffusions are given. After this the joint distribution of g_0 and d_0 is derived. In particular, it is seen that this distribution belongs to a special class of two-dimensional distributions characterized by the fact that the density (when it exists) is a function of the sum of the arguments only. The proof of the main identity (1.2) is presented in Section 2.3. After this, in Section 2.4 we give an alternative proof of (1.2) for RBM with drift based on the Ray–Knight theorems. In the second main section the corresponding identity for the excursion bridges is discussed. Here we start with by relating the distribution of $(-g_0, d_0)$ to the Lévy measure of the inverse local time preparing in this way the connection of the Itô excursion measure and the Palm measure. In Section 3.2 we make some observations concerning the spectral representations of d_0 and $d_0 - g_0$. The result concerning excursion bridges is proved in Section 3.3 by utilizing the connections of the Itô excursion measure and the Palm measure. In Section 4 we discuss the null recurrent case and, finally, in Section 5 we give some examples.

2 Stationary excursions straddling t

2.1 Preliminaries

Let $X^\circ = \{X^\circ_t : t \geq 0\}$ be a one-dimensional recurrent conservative diffusion living in the interval I. We assume that $I = [0, b]$, $b < +\infty$, or $I = [0, b)$, $b \leq +\infty$, where 0 is a reflecting boundary and b is either reflecting or natural or entrance-not-exit. For background on one-dimensional diffusions we refer to [13] and [6].

The diffusion X° is characterized in I by its scale function $S(x)$ and speed measure $m(dx)$. Suppose, moreover, that X° is positively recurrent, that is $M := m\{I\} < \infty$. Let $p(t; x, y)$ be the symmetric transition density of X° with respect to $m(dx)$ and let

$$\mathcal{G} = \frac{d}{dm}\frac{d}{dS}$$

be the infinitesimal generator of X. Assume that $m(dx) = m(x)dx$, $S(dx) = S'(x)dx$ such that $m(x)$ and $S'(x)$ are continuous and positive. The Green function is defined as

$$G_\alpha(x, y) := \int_0^\infty e^{-\alpha t} p(t; x, y)\, dt.$$

Let $\mathbf{P}_x$ and $\mathbf{E}_x$ denote the probability measure and the expectation, respectively, associated with X° started at x and $H_y(X^\circ)$ denote the first hitting time of y for X°. Recall from [13] p. 129 that

$$\mathbf{E}_x\left(e^{-\alpha H_y(X^\circ)}\right) = \frac{G_\alpha(x, y)}{G_\alpha(y, y)}. \tag{2.1}$$

Let now $\{X^{(1)}_t : t \geq 0\}$ and $\{X^{(2)}_t : t \geq 0\}$ be two copies of X° such that $X^{(1)}_0 = X^{(2)}_0$ with the common law $\widetilde{m}(dx) := m(dx)/M$ but let $X^{(1)}$ and $X^{(2)}$ be otherwise independent. Define for $t \in \mathbb{R}$,

$$X_t := \begin{cases} X^{(1)}_t, & t \geq 0, \\ X^{(2)}_{-t}, & t \leq 0. \end{cases}$$

The process $X = \{X_t : t \in \mathbb{R}\}$ is called a stationary diffusion in stationary state living in I and having the generator $\mathcal{G}$. Notice that the law of X_t is $\widetilde{m}$ for every $t \in \mathbb{R}$.

2.2 Joint distribution of g_0 and d_0

For the process X defined above let g_0 and d_0 be as in (1.1). In this section we derive the joint distribution of $-g_0$ and d_0 and show that the pair $(-g_0, d_0)$ is an element of a special class $\mathcal{K}$ of two-dimensional random variables studied in [30] and [17]. We start with the definition and some properties of this class $\mathcal{K}$.

Definition 2.1. A two-dimensional random variable (ξ_1, ξ_2), where both ξ_1 and ξ_2 are non-negative, belongs to $\mathcal{K}$ if

$$(\xi_1, \xi_2) \stackrel{d}{=} (U, V - U),$$

where V is an arbitrary non-negative random variable and the conditional distribution of U given V is uniform on $(0, V)$.

Proposition 2.2. (Properties of the elements in $\mathcal{K}$)
1) Let $(\xi_1, \xi_2) \in \mathcal{K}$. Then $\xi_1 \stackrel{d}{=} \xi_2$, $\xi_1 + \xi_2 \stackrel{d}{=} V$, and the density of ξ_1 (and ξ_2) exists and is given by

$$F'_{\xi_1}(x) = \int_{[x,\infty)} v^{-1} F_V(dv).$$

If the density $F'_V(v) =: p(v)$ exists then

$$\mathbf{P}(\xi_1 \in dx, \xi_2 \in dy) = \frac{p(x+y)}{x+y}\, dx\, dy. \tag{2.2}$$

2) Let ξ_1 and ξ_2 be non-negative random variables. Then $(\xi_1, \xi_2) \in \mathcal{K}$ if and only if for all $\alpha \neq \beta$,

$$\mathbf{E}\left(e^{-\alpha\xi_1 - \beta\xi_2}\right) = \frac{1}{\alpha - \beta} \int_\beta^\alpha \mathbf{E}\big(e^{-\gamma(\xi_1+\xi_2)}\big)\, d\gamma. \tag{2.3}$$

Proof: For the proof, see [30] and [17]. □

Proposition 2.3. *The pair $(-g_0, d_0)$ belongs to the class $\mathcal{K}$ with the joint Laplace transform given by*

$$\mathbf{E}\left(e^{-\alpha d_0 + \beta g_0}\right) = \frac{1}{M(\alpha - \beta)} \left(\frac{1}{G_\alpha(0,0)} - \frac{1}{G_\beta(0,0)} \right). \tag{2.4}$$

Moreover, the joint density of $(-g_0, d_0)$ *exists and is given by*

$$\begin{aligned}\mathbf{P}(d_0 \in dt, -g_0 \in ds) \\ = \frac{1}{M}\left(\frac{d}{dS(y_1)}\frac{d}{dS(y_2)}\widehat{p}(t+s;y_1,y_2)\right)\Big|_{y_1,y_2=0+} dt\,ds, \qquad (2.5)\end{aligned}$$

where $\widehat{p}(t;x,y)$ *is the transition density of the process* $\widehat{X}$ *obtained from* X° *by killing* X° *when it hits zero.*

Proof: Using conditional independence of $X^{(1)}$ and $X^{(2)}$, formula (2.1), and the Chapman-Kolmogorov equation, we write

$$\begin{aligned}
\mathbf{E}\left(e^{-\alpha d_0+\beta g_0}\right) &= \int_I \widetilde{m}(dx)\,\mathbf{E}_x\left(e^{-\alpha H_0(X^{(1)})}\right)\mathbf{E}_x\left(e^{-\beta H_0(X^{(2)})}\right) \\
&= \int_I \widetilde{m}(dx)\,G_\alpha(x,0)G_\beta(x,0)\,/(G_\alpha(0,0)G_\beta(0,0)) \\
&= \int_I \widetilde{m}(dx)\int_0^\infty dt\,e^{-\alpha t}p(t;x,0) \\
&\qquad \times \int_0^\infty ds\,e^{-\beta s}p(s;x,0)/(G_\alpha(0,0)G_\beta(0,0)) \\
&= \int_0^\infty dt\int_0^\infty ds\,e^{-\alpha t-\beta s} \\
&\qquad \times \int_I \widetilde{m}(dx)\,p(t;x,0)p(s;x,0)/(G_\alpha(0,0)G_\beta(0,0)) \\
&= \frac{1}{M}\int_0^\infty dt\int_0^\infty ds\,e^{-\alpha t-\beta s}p(t+s;0,0)\,/(G_\alpha(0,0)G_\beta(0,0)) \\
&= \frac{1}{M(\alpha-\beta)}\left(\frac{1}{G_\alpha(0,0)}-\frac{1}{G_\beta(0,0)}\right).
\end{aligned}$$

To show that $(-g_0, d_0) \in \mathcal{K}$ we use Proposition 2.2. Letting $\alpha \to \beta$ in (2.4), it is seen that the derivative $\frac{d}{d\alpha}\left(\frac{1}{G_\alpha(0,0)}\right)$ exists and

$$\begin{aligned}
\mathbf{E}\left(e^{-\alpha d_0+\beta g_0}\right) &= \frac{1}{M(\alpha-\beta)}\int_\beta^\alpha \frac{d}{d\gamma}\left(\frac{1}{G_\gamma(0,0)}\right)d\gamma \\
&= \frac{1}{M(\alpha-\beta)}\int_\beta^\alpha \mathbf{E}\left(e^{-\gamma(d_0-g_0)}\right)d\gamma.
\end{aligned}$$

To compute the joint density recall the formula (see [13], p. 154)

$$\mathbf{P}_x(H_0(X^\circ)\in dt)/dt = \frac{d\widehat{p}(t;x,y)}{dS(y)}\Big|_{y=0+} =: \widehat{p}^+(t;x,0) \qquad (2.6)$$

and, hence,

$$
\begin{aligned}
&\mathbf{P}(d_0 \in dt, -g_0 \in ds)/dt\,ds \\
&\quad = \int_I \widetilde{m}(dx)\widehat{p}^{+}(t;x,0)\widehat{p}^{+}(s;x,0) \\
&\quad = \frac{1}{M}\left(\frac{d}{dS(y_1)}\frac{d}{dS(y_2)}\int_I m(dx)\,\widehat{p}(t;x,y_1)\widehat{p}(s;x,y_2)\right)\Big|_{y_1,y_2=0+} \\
&\quad = \frac{1}{M}\left(\frac{d}{dS(y_1)}\frac{d}{dS(y_2)}\widehat{p}(t+s;y_1,y_2)\right)\Big|_{y_1,y_2=0+},
\end{aligned}
$$

where it is used that $\widehat{p}(t;x,y)$ is continuous in t, x, y and that $\widehat{p}^{+}(t;x,y)$ is continuous in t, x and right continuous in y (see [13] p. 149). □

Remark 2.4. To find the density (2.5) we used formula (2.6) and the Chapman-Kolmogorov equation. There is also an alternative approach. For this, let $\widehat{G}_\alpha(x,y)$ be the Green function for the killed process $\widehat{X}$. Then (cf. [8] p. 185)

$$
\frac{d}{dS(x)}\frac{d}{dS(y)}\widehat{G}_\alpha(x,y)\Big|_{x,y=0+} = -\frac{1}{G_\alpha(0,0)} \tag{2.7}
$$

and we have

$$
\begin{aligned}
&\frac{1}{\alpha-\beta}\left(\frac{1}{G_\alpha(0,0)} - \frac{1}{G_\beta(0,0)}\right) \\
&\qquad = \frac{1}{\alpha-\beta}\frac{d}{dS(x)}\frac{d}{dS(y)}\left(\widehat{G}_\beta(x,y) - \widehat{G}_\alpha(x,y)\right)\Big|_{x,y=0+} \\
&\qquad = \frac{1}{\alpha-\beta}\int_0^\infty dt\ \left(e^{-\beta t} - e^{-\alpha t}\right)\frac{d}{dS(x)}\frac{d}{dS(y)}\widehat{p}(t;x,y)\big|_{x,y=0+} \\
&\qquad = \int_0^\infty\int_0^\infty e^{-\alpha t-\beta s}\frac{d}{dS(x)}\frac{d}{dS(y)}\widehat{p}(t+s;x,y)\big|_{x,y=0+}\,dt\,ds.
\end{aligned}
$$

2.3 Main identity in law

The first main result of the paper is presented in the following theorem.

Theorem 2.5. *Let X be a stationary diffusion in stationary state. Then for all $t \in \mathbb{R}$, the two-dimensional random variables $(t-g_t, d_t-t)$ and $(I_t^{(+)}, I_t^{(-)})$ are identically distributed and, from Proposition 2.3,*

$$
\mathbf{E}\big(e^{-\alpha I_t^{(+)} - \beta I_t^{(-)}}\big) = \frac{1}{M(\alpha-\beta)}\left(\frac{1}{G_\alpha(0,0)} - \frac{1}{G_\beta(0,0)}\right), \tag{2.8}
$$

$$
\begin{aligned}
&\mathbf{P}\big(I_t^{(+)} \in dr, I_t^{(-)} \in ds\big) \\
&\qquad = \frac{1}{M}\left(\frac{d}{dS(y_1)}\frac{d}{dS(y_2)}\widehat{p}(s+r;\, y_1, y_2)\right)\Big|_{y_1,y_2=0+} dr\, ds. \qquad (2.9)
\end{aligned}
$$

Proof: We prove the theorem only for the case when the state space is $I = [0, +\infty)$ and leave other cases to the reader.

By the stationarity, it suffices to consider the case $t = 0$. To simplify the notation, let $I^{(+)} := I_0^{(+)}$ and $I^{(-)} := I_0^{(-)}$. For $y \in (0, \infty)$ introduce

$$
u(x) := \mathbf{E}_x\left(\exp\Big(-\alpha\int_0^{H_0(X)} \mathbf{1}_{\{0\le X_s\le y\}} ds - \beta\int_0^{H_0(X)} \mathbf{1}_{\{X_s>y\}} ds\Big)\right).
$$

From the Feynman-Kac formula it follows that $u(x)$ is the unique bounded smooth solution of the equation

$$
\mathcal{G}u(x) = \begin{cases} \alpha\, u(x), & 0 < x < y, \\ \beta\, u(x), & x > y \end{cases}
$$

with the boundary condition $u(0) = 1$. The function $u(x)$, therefore, has the following form

$$
u(x) := \begin{cases} A\widehat{\psi}_\alpha(x) + \widehat{\varphi}_\alpha(x)/\widehat{\varphi}_\alpha(0), & x \le y, \\ B\widehat{\varphi}_\beta(x), & x \ge y, \end{cases}
$$

for some constants A and B, where $\widehat{\psi}_\alpha$ and $\widehat{\varphi}_\alpha$ are the increasing and decreasing fundamental solution, respectively, of the generalized differential equation

$$
\mathcal{G}u = \alpha u \tag{2.10}
$$

such that $\widehat{\varphi}_\alpha$ is bounded and the "killing" condition holds

$$
\widehat{\psi}_\alpha(0) = 0. \tag{2.11}
$$

Both u and its derivative u^+ should be continuous at y, i.e.,

$$
\begin{cases} A\widehat{\psi}_\alpha(y) + \widehat{\varphi}_\alpha(y)/\widehat{\varphi}_\alpha(0) = B\widehat{\varphi}_\beta(y) \\ A\widehat{\psi}_\alpha^+(y) + \widehat{\varphi}_\alpha^+(y)/\widehat{\varphi}_\alpha(0) = B\widehat{\varphi}_\beta^+(y). \end{cases} \tag{2.12}
$$

From (2.12),

$$\begin{aligned} B &= \frac{\widehat{\psi}^+_\alpha(y)\widehat{\varphi}_\alpha(y) - \widehat{\psi}_\alpha(y)\widehat{\varphi}^+_\alpha(y)}{\widehat{\varphi}_\alpha(0)\big(\widehat{\psi}^+_\alpha(y)\widehat{\varphi}_\beta(y) - \widehat{\psi}_\alpha(y)\widehat{\varphi}^+_\beta(y)\big)} \\ &= \frac{\widehat{\psi}^+_\alpha(0)}{\widehat{\psi}^+_\alpha(y)\widehat{\varphi}_\beta(y) - \widehat{\psi}_\alpha(y)\widehat{\varphi}^+_\beta(y)}, \end{aligned}$$

where it is used that the Wronskian (a constant)

$$\widehat{w}_\alpha := \widehat{\psi}^+_\alpha(x)\widehat{\varphi}_\alpha(x) - \widehat{\psi}_\alpha(x)\widehat{\varphi}^+_\alpha(x) \tag{2.13}$$

equals $\widehat{\psi}^+_\alpha(0)\widehat{\varphi}_\alpha(0)$. Consequently, at point y,

$$u(y) = \frac{\widehat{\psi}^+_\alpha(0)\widehat{\varphi}_\beta(y)}{\widehat{\psi}^+_\alpha(y)\widehat{\varphi}_\beta(y) - \widehat{\psi}_\alpha(y)\widehat{\varphi}^+_\beta(y)}.$$

The joint Laplace transform of $I^{(-)}$ and $I^{(+)}$ is given then as

$$\begin{aligned} \mathbf{E}\big(e^{-\alpha I^{(-)}-\beta I^{(+)}}\big) &= \int_0^\infty \widetilde{m}(dy)\,(u(y))^2 \\ &= \big(\widehat{\psi}^+_\alpha(0)\big)^2 \int_0^\infty \widetilde{m}(dy)\left(\frac{\widehat{\varphi}_\beta(y)}{\widehat{\psi}^+_\alpha(y)\widehat{\varphi}_\beta(y) - \widehat{\psi}_\alpha(y)\widehat{\varphi}^+_\beta(y)}\right)^2. \end{aligned} \tag{2.14}$$

Let

$$r(y) := \widehat{\psi}^+_\alpha(y)\widehat{\varphi}_\beta(y) - \widehat{\psi}_\alpha(y)\widehat{\varphi}^+_\beta(y)$$

denote the denominator in the expression for $u(y)$ and notice that

$$\begin{aligned} \frac{d}{dm}r(y) &= \frac{d}{dm}\left(\widehat{\varphi}_\beta(y)\frac{d}{dS}\widehat{\psi}_\alpha(y) - \widehat{\psi}_\alpha(y)\frac{d}{dS}\widehat{\varphi}_\beta(y)\right) \\ &= (\alpha-\beta)\widehat{\psi}_\alpha(y)\widehat{\varphi}_\beta(y), \end{aligned} \tag{2.15}$$

where we used that $m(dx) = m(x)dx$ and $S(dx) = S'(x)dx$ and that $\widehat{\psi}_\alpha(x)$ and $\widehat{\varphi}_\beta(x)$ satisfy the equations

$$\mathcal{G}\widehat{\psi}_\alpha = \alpha\widehat{\psi}_\alpha \quad \text{and} \quad \mathcal{G}\widehat{\varphi}_\beta = \beta\widehat{\varphi}_\beta,$$

respectively. The equality (2.15) is the key to obtain

$$
\begin{aligned}
\mathbf{E}\big(e^{-\alpha I^{(-)}-\beta I^{(+)}}\big) &= \big(\widehat{\psi}_\alpha^+(0)\big)^2 \int_0^\infty \widetilde{m}(dy)\frac{\big(\widehat{\varphi}_\beta(y)\big)^2}{\big(r(y)\big)^2} \\
&= \frac{\big(\widehat{\psi}_\alpha^+(0)\big)^2}{M(\alpha-\beta)} \int_0^\infty m(dy)\frac{\widehat{\varphi}_\beta(y)}{\widehat{\psi}_\alpha(y)}\frac{\frac{d}{dm}r(y)}{\big(r(y)\big)^2} \\
&= \frac{\big(\widehat{\psi}_\alpha^+(0)\big)^2}{M(\alpha-\beta)} \lim_{\varepsilon\to 0}\left\{\int_\varepsilon^\infty m(dy)\frac{\widehat{\varphi}_\beta(y)}{\widehat{\psi}_\alpha(y)}\frac{\frac{d}{dm}r(y)}{\big(r(y)\big)^2}\right\}.
\end{aligned}
$$

Let $F(\varepsilon)$ denote the integral above. Using that $\widehat{\varphi}_\beta(x)$ is bounded and $\widehat{\psi}_\alpha(x) \to +\infty$ as $x \to +\infty$, we have

$$
\begin{aligned}
F(\varepsilon) &= \int_\varepsilon^\infty m(dy)\frac{\widehat{\varphi}_\beta(y)}{\widehat{\psi}_\alpha(y)}\frac{d}{dm}\left(-\frac{1}{r(y)}\right) \\
&= \int_\varepsilon^\infty m(dy)\frac{d}{dm}\left(\frac{1}{r(y)}\right)\int_y^\infty S(dx)\frac{d}{dS}\left(\frac{\widehat{\varphi}_\beta(x)}{\widehat{\psi}_\alpha(x)}\right) \\
&= \int_\varepsilon^\infty S(dx)\frac{d}{dS}\left(\frac{\widehat{\varphi}_\beta(x)}{\widehat{\psi}_\alpha(x)}\right)\int_\varepsilon^x m(dy)\frac{d}{dm}\left(\frac{1}{r(y)}\right) \\
&= \int_\varepsilon^\infty S(dx)\frac{d}{dS}\left(\frac{\widehat{\varphi}_\beta(x)}{\widehat{\psi}_\alpha(x)}\right)\left(\frac{1}{r(x)}-\frac{1}{r(\varepsilon)}\right).
\end{aligned}
$$

Since

$$
r(x) = -\big(\widehat{\psi}_\alpha(x)\big)^2\frac{d}{dS}\left(\frac{\widehat{\varphi}_\beta(x)}{\widehat{\psi}_\alpha(x)}\right),
$$

we obtain

$$
F(\varepsilon) = -\int_\varepsilon^\infty \frac{S(dx)}{\big(\widehat{\psi}_\alpha(x)\big)^2} + \frac{\widehat{\varphi}_\beta(\varepsilon)}{\widehat{\psi}_\alpha(\varepsilon)r(\varepsilon)}.
$$

By definition (2.13) of the Wronskian,

$$
\frac{1}{\widehat{w}_\alpha}\frac{d}{dS}\left(\frac{\widehat{\varphi}_\alpha(x)}{\widehat{\psi}_\alpha(x)}\right) = -\frac{1}{\big(\widehat{\psi}_\alpha(x)\big)^2},
$$

and, hence,

$$\begin{aligned}
F(\varepsilon) &= -\frac{1}{\widehat{w}_\alpha}\frac{\widehat{\varphi}_\alpha(\varepsilon)}{\widehat{\psi}_\alpha(\varepsilon)} + \frac{\widehat{\varphi}_\beta(\varepsilon)}{\widehat{\psi}_\alpha(\varepsilon)\big(\widehat{\psi}^+_\alpha(\varepsilon)\widehat{\varphi}_\beta(\varepsilon) - \widehat{\psi}_\alpha(\varepsilon)\widehat{\varphi}^+_\beta(\varepsilon)\big)} \\
&= \frac{1}{\widehat{\psi}_\alpha(\varepsilon)}\left(\frac{-\widehat{\varphi}_\alpha(\varepsilon)}{\widehat{\psi}^+_\alpha(\varepsilon)\widehat{\varphi}_\alpha(\varepsilon) - \widehat{\psi}_\alpha(\varepsilon)\widehat{\varphi}^+_\alpha(\varepsilon)} + \frac{\widehat{\varphi}_\beta(\varepsilon)}{\widehat{\psi}^+_\alpha(\varepsilon)\widehat{\varphi}_\beta(\varepsilon) - \widehat{\psi}_\alpha(\varepsilon)\widehat{\varphi}^+_\beta(\varepsilon)}\right) \\
&= \frac{\widehat{\varphi}_\alpha(\varepsilon)\widehat{\varphi}^+_\beta(\varepsilon) - \widehat{\varphi}_\beta(\varepsilon)\widehat{\varphi}^+_\alpha(\varepsilon)}{\widehat{w}_\alpha\big(\widehat{\psi}^+_\alpha(\varepsilon)\widehat{\varphi}_\beta(\varepsilon) - \widehat{\psi}_\alpha(\varepsilon)\widehat{\varphi}^+_\beta(\varepsilon)\big)}.
\end{aligned}$$

Consequently, using boundary condition (2.11),

$$\begin{aligned}
\mathbf{E}\big(e^{-\alpha I^{(-)} - \beta I^{(+)}}\big) &= \frac{\big(\widehat{\psi}^+_\alpha(0)\big)^2}{M(\alpha-\beta)}\lim_{\varepsilon\to 0} F(\varepsilon) \\
&= \frac{\big(\widehat{\psi}^+_\alpha(0)\big)^2}{M(\alpha-\beta)}\,\frac{\big(\widehat{\varphi}_\alpha(0)\widehat{\varphi}^+_\beta(0) - \widehat{\varphi}_\beta(0)\widehat{\varphi}^+_\alpha(0)\big)}{\widehat{w}_\alpha\widehat{\psi}^+_\alpha(0)\widehat{\varphi}_\beta(0)} \\
&= \frac{1}{M(\alpha-\beta)}\left(\frac{\widehat{\varphi}^+_\beta(0)}{\widehat{\varphi}_\beta(0)} - \frac{\widehat{\varphi}^+_\alpha(0)}{\widehat{\varphi}_\alpha(0)}\right). \qquad (2.16)
\end{aligned}$$

Finally, the Green function of X can be represented as

$$G_\alpha(x,y) = w_\alpha^{-1}\psi_\alpha(x)\varphi_\alpha(y), \quad x \le y$$

(see, e.g., [6] p. 19), where ψ_α, φ_α are the fundamental solutions of (2.10) such that φ_α is bounded and $\psi^+_\alpha(0) = 0$ (condition for the reflection). Since $\widehat{\varphi}_\alpha = \varphi_\alpha$, we have

$$\frac{1}{G_\alpha(0,0)} = -\frac{\varphi^+_\alpha(0)}{\varphi_\alpha(0)} = -\frac{\widehat{\varphi}^+_\alpha(0)}{\widehat{\varphi}_\alpha(0)}.$$

Thus, the right-hand sides of (2.16) and (2.4) are equal and the proof is complete. □

Although we do not have any probabilistic explanation for the equality in law in Theorem 2.5 (but for RBM with drift, see the next section), the following lemma (cf. [29], Proposition 3.11 p. 330) explains why the variables $I^{(+)}_t$, $I^{(-)}_t$, $t - g_t$, and $d_t - t$ all have the same expectation.

Lemma 2.6. *For a given $s > 0$ the events $\{X_s > X_0\}$ and $\{d_0 > s\}$ are independent.*

Proof: By the symmetry of the transition densities,

$$\begin{aligned}
\mathbf{P}(X_s > X_0\,;\, s < d_0) &= \int_0^b \frac{m(dx)}{M} \mathbf{P}_x(X_s^{(1)} > x\,;\, s < H_0(X^{(1)})) \\
&= \frac{1}{M} \int_0^b m(dx) \int_x^b m(dy)\, \widehat{p}(s; x, y) \\
&= \int_0^b \frac{m(dy)}{M} \int_0^y m(dx)\, \widehat{p}(s; y, x) \\
&= \mathbf{P}(X_s < X_0\,;\, s < d_0). \qquad (2.17)
\end{aligned}$$

Noting that, by the reversibility and the stationarity,

$$\mathbf{P}(X_s > X_0) = \mathbf{P}(X_{-s} > X_0) = \mathbf{P}(X_0 > X_s) = 1/2$$

completes the proof. □

Corollary 2.7. *The expectations of $I^{(+)}$, $I^{(-)}$, and d_0 are equal.*

Proof: By (2.17),

$$\begin{aligned}
\mathbf{E}\left(I^{(+)}\right) &= 2\,\mathbf{E}\Big(\int_0^{d_0} \mathbf{1}_{\{X_s > X_0\}}\, ds\Big) = 2\int_0^\infty \mathbf{P}(X_s > X_0\,;\, s < d_0)\, ds \\
&= \int_0^\infty \mathbf{P}(d_0 > s)\, ds = \mathbf{E}(d_0).
\end{aligned}$$

Similarly, we can show that $\mathbf{E}\left(I^{(-)}\right) = \mathbf{E}(d_0)$. □

2.4 Probabilistic explanation of the identity when X is a RBM with drift

In the case when X is a stationary reflecting Brownian motion with drift $-\mu < 0$ we prove the fact that $I^{(+)} \stackrel{d}{=} I^{(-)} \stackrel{d}{=} d_0$ by considering the local time process of an excursion instead of applying the Feynman-Kac formula as in the proof of Theorem 2.5.

Let $\{L(t, y) : t \geq 0, y \geq 0\}$ denote the local time of $\{X_s^\circ : s \geq 0\}$ up to t at the level y with respect to the Lebesgue measure, i.e., we have

$$\int_0^t g(X_s^\circ)\, ds = \int_E g(x) L(t, x)\, dx \qquad (2.18)$$

for any non-negative Borel function g. The following Ray–Knight theorem (see [6] pp. 90–91) describes the behaviour of the local time process L up to $H_0(X)$. Let $Z^{(n,2\mu)}$ denote the squared radial part of an n-dimensional Ornstein–Uhlenbeck process with parameter μ and recall that the generator of $Z^{(n,2\mu)}$ is

$$2z\frac{d^2}{dz^2}+(n-2\mu z)\frac{d}{dz}.$$

Theorem 2.8. *Conditionally on $X_0=x$,*

$$\{L(H_0(X),y):0\le y\le x\}\stackrel{d}{=}\{Z_y^{(2,2\mu)}:0\le y\le x\},$$

$$\{L(H_0(X),x+y):y\ge 0\}\stackrel{d}{=}\{Z_y^{(0,2\mu)}:y\ge 0\},$$

where the process $Z^{(0,2\mu)}$ is started from the position of $Z^{(2,2\mu)}$ at time x but otherwise $Z^{(2,2\mu)}$ and $Z^{(0,2\mu)}$ are independent.

For the stationary excursion $\{X_t:g_0<t<d_0\}$ straddling zero introduce its total local time process by

$$L^{(e)}(y):=L^{(1)}(H_0(X^{(1)}),y)+L^{(2)}(H_0(X^{(2)}),y),\quad y\ge 0,$$

where $L^{(1)}(H_0(X^{(1)}),y)$ and $L^{(2)}(H_0(X^{(2)}),y)$ are the local times at y up to the first hitting times of zero for $X^{(1)}$ and $X^{(2)}$, respectively. Since $X^{(1)}$ and $X^{(2)}$ are independent, given $X_0=x$, it follows that $L^{(1)}$ and $L^{(2)}$ are also independent, given $X_0=x$.

Recall (see [32] and [6] p. 72) that if $Y^{(1)}$ and $Y^{(2)}$ are two independent non-negative time-homogeneous diffusions then $Y^{(1)}+Y^{(2)}$ is a diffusion if and only if the generators of $Y^{(1)}$ and $Y^{(2)}$ are of the forms

$$ax\frac{d^2}{dx^2}+(bx+c_i)\frac{d}{dx},\quad i=1,2,$$

respectively, where $a>0$, $c_i\ge 0$, $b\in\mathbb{R}$. The generator of $Y^{(1)}+Y^{(2)}$ is

$$ax\frac{d^2}{dx^2}+(bx+(c_1+c_2))\frac{d}{dx}.$$

From Theorem 2.8 we obtain now the Ray–Knight theorem for $L^{(e)}$.

Theorem 2.9. *Conditionally on $X_0 = x$,*

$$\{L^{(e)}(y) : 0 \le y \le x\} \stackrel{d}{=} \{Z_y^{(4,2\mu)} : 0 \le y \le x\},$$

$$\{L^{(e)}(x+y) : y \ge 0\} \stackrel{d}{=} \{Z_y^{(0,2\mu)} : y \ge 0\},$$

where the process $Z^{(4,2\mu)}$ is started from 0 and the process $Z^{(0,2\mu)}$ is started from the position of $Z^{(4,2\mu)}$ at time x but otherwise $Z^{(4,2\mu)}$ and $Z^{(0,2\mu)}$ are independent.

Lemma 2.10. *Let $H_0(L) := \inf\{y \ge 0 : L^{(e)}(X_0+y) = 0\}$. Then the random variables $H_0(L)$ and $L^{(e)}(X_0)$ are exponentially distributed with parameters 2μ and μ, respectively.*

Proof: Define

$$M^{(1)} := \sup\{X_t :\ 0 \le t \le d_0\} \text{ and } M^{(2)} := \sup\{X_t :\ g_0 \le t \le 0\}.$$

The distribution of $M^{(i)}$ $(i = 1, 2)$ is given by (see [6] p. 14)

$$\mathbf{P}_x(M^{(i)} < y) = \mathbf{P}_x(H_0(X) < H_y(X)) = \frac{S(y) - S(x)}{S(y) - S(0)}.$$

Let $\widetilde{M}^{(i)} := M^{(i)} - x$. Then

$$\mathbf{P}_x(\widetilde{M}^{(i)} < y) = \mathbf{P}_x(M^{(i)} < x + y) = \frac{e^{2\mu(x+y)} - e^{2\mu x}}{e^{2\mu(x+y)} - 1},$$

where we used that for RBM with drift

$$S(x) = \frac{1}{2\mu}(1 - e^{2\mu x}).$$

Hence, by integration,

$$\begin{aligned} \mathbf{P}(H_0(L) < y) &= \int_0^\infty 2\mu e^{-2\mu x}\, \mathbf{P}_x(\widetilde{M}^{(1)} < y)\mathbf{P}_x(\widetilde{M}^{(2)} < y)\, dx \\ &= \int_0^\infty 2\mu e^{-2\mu x} \left(\frac{e^{2\mu(x+y)} - e^{2\mu x}}{e^{2\mu(x+y)} - 1}\right)^2 dx \\ &= 1 - e^{-2\mu y}. \end{aligned}$$

Next we show that $L^{(e)}(X_0) \sim Exp(\mu)$, that is $L^{(e)}(X_0)$ is exponentially distributed with parameter μ. Recall that for Brownian motion with drift $-\mu$ killed at zero

$$\widehat{G}_0(x,x) = \frac{1}{2\mu}\left(e^{2\mu x} - 1\right).$$

The Laplace transform of $L(H_0, x)$ is (see [6] p. 32)

$$\mathbf{E}_x\left(e^{-\alpha L(H_0(X),x)}\right) = \frac{1}{\widehat{G}_0(x,x)m(x)\alpha + 1} = \frac{\mu}{(1-e^{-2\mu x})\alpha + \mu}. \tag{2.19}$$

From (2.19) we find

$$\mathbf{E}\big(e^{-\alpha L^{(e)}(X_0)}\big) = \int_0^\infty 2\mu e^{-2\mu x}\Big(\frac{\mu}{(1-e^{-2\mu x})\alpha+\mu}\Big)^2 dx = \frac{\mu}{\mu+\alpha}.$$

□

Proposition 2.11. *The random variables $I^{(+)}$, $I^{(-)}$, and d_0 have the same law.*

Proof: We show first that $I^{(+)} \stackrel{d}{=} I^{(-)}$. By Proposition 2.9 and (2.18),

$$\begin{aligned}
\mathbf{E}\big(e^{-\alpha I^{(+)}}\big) &= \mathbf{E}\Big(\exp\Big(-\alpha\int_0^\infty \mathbf{1}_{\{X_s > X_0\}} ds\Big)\Big) \\
&= \mathbf{E}\Big(\exp\Big(-\alpha\int_0^\zeta Z_y^{(0,2\mu)}\, dy\Big)\Big),
\end{aligned}$$

where $Z_0^{(0,2\mu)} \sim Exp(\mu)$ and $\zeta \sim Exp(2\mu)$ is the life time of $Z^{(0,2\mu)}$. Note that (cf. [6] p. 90)

$$\{Z^{(0,2\mu)}(\zeta - y) : 0 \le y \le \zeta\} \stackrel{d}{=} \{Z^{(4,2\mu)}(y) : 0 \le y \le \tau\},$$

where $\tau \sim Exp(2\mu)$ is independent of $Z^{(4,2\mu)}$. Consequently, by Proposition 2.9,

$$\begin{aligned}
\mathbf{E}\big(e^{-\alpha I^{(+)}}\big) &= \mathbf{E}\Big(\exp\Big(-\alpha\int_0^\zeta Z_y^{(0,2\mu)}\, dy\Big)\Big) \\
&= \mathbf{E}\Big(\exp\Big(-\alpha\int_0^\zeta Z_{(\zeta-y)}^{(0,2\mu)}\, dy\Big)\Big) \\
&= \mathbf{E}\Big(\exp\Big(-\alpha\int_0^\tau Z_y^{(4,2\mu)}\, dy\Big)\Big) \\
&= \mathbf{E}\big(e^{-\alpha I^{(-)}}\big).
\end{aligned}$$

To deduce that $I^{(+)} \stackrel{d}{=} d_0$, we adopt the argument in [31] used in the proof of the Biane–Imhof identity. First recall that

$$I^{(+)} = \int_0^\infty Z_s\, ds,$$

where the process $Z = Z^{(0,2\mu)}$ is the solution of the SDE

$$dZ_t = 2\sqrt{Z_t}\, dW_t - 2\mu\, Z_t\, dt, \quad Z_0 \sim Exp(\mu). \tag{2.20}$$

Introduce next

$$A_t := \int_0^t Z_s\, ds$$

and let α_t be the right-continuous inverse of A_t. Since the quadratic variation of the local martingale $Y_t = \int_0^t \sqrt{Z_s}\, dW_s$ is given by

$$\langle Y, Y\rangle_t = \int_0^t Z_s\, ds = A_t,$$

it follows from Lévy's characterization theorem that Y_{α_t} is a Brownian motion, say B_t, started at zero and stopped at A_∞. Hence, for $t < A_\infty$,

$$\begin{aligned} Z_{\alpha_t} - Z_0 &= 2\int_0^{\alpha_t} \sqrt{Z_s}\, dW_s - 2\mu \int_0^{\alpha_t} Z_s\, ds \\ &= 2Y_{\alpha_t} - 2\mu A_{\alpha_t} \\ &= 2B_t - 2\mu t. \end{aligned}$$

Letting $t \to A_\infty$ gives $Z_0/2 = B^{(\mu)}_{A_\infty}$, where $B^{(\mu)}_t := -B_t + \mu t$. Thus, taking into account that

$$0 < Z_{\alpha_t} = Z_0 + 2B_t - 2\mu t, \quad 0 \le t < A_\infty,$$

yields

$$A_\infty = \inf\{t : B^{(\mu)}_t = Z_0/2\}. \tag{2.21}$$

Since $Z_0 \sim Exp(\mu)$ and is independent of $B^{(\mu)}$, (2.21) is equivalent with

$$I^+ = A_\infty \stackrel{d}{=} H_\lambda(B^{(\mu)}), \tag{2.22}$$

where $\lambda \sim Exp(2\mu)$ is independent of $B^{(\mu)}$. Noting that the right-hand side of (2.22) is identical in law to $H_0(X)$ gives $I^{(+)} \stackrel{d}{=} d_0$, as claimed. $\square$

3 Relation to Itô excursion theory

3.1 Distribution of $(-g_0, d_0)$ in terms of Lévy measure

Consider the distribution of d_0 (and $-g_0$). From (2.4),

$$\mathbf{E}\left(e^{-\alpha d_0}\right) = \mathbf{E}\left(e^{\alpha g_0}\right) = \frac{1}{M\alpha G_\alpha(0,0)}. \tag{3.1}$$

Next let $\ell(t,0)$ be the local time at zero up to time t (with respect to the speed measure) of $\{X_s : s \geq 0\}$, $X_0 = 0$. Let $A = \{A_t : t \geq 0\}$ be the right-continuous inverse of ℓ. The process A is a subordinator and

$$\mathbf{E}_0\big(\exp(-\alpha A_t)\big) = \exp(-t\,\Psi(\alpha)),$$

where the Laplace exponent Ψ is given by

$$\Psi(\alpha) = \int_0^\infty \left(1 - e^{-\alpha t}\right) n^+(dt) = \alpha \int_0^\infty e^{-\alpha t} n^+(t,\infty)\, dt$$

with the associated Lévy measure $n^+(dt)$.

Proposition 3.1. *The Laplace exponent $\Psi(\alpha)$ of A is given by*

$$\Psi(\alpha) = \frac{1}{G_\alpha(0,0)} = -\frac{d}{dS(x)}\mathbf{E}_x\left(e^{-\alpha H_0}\right)\Big|_{x=0+}. \tag{3.2}$$

Proof: See [13] p. 214. □

Proposition 3.2. *Let $n^+(dt)$ be the Lévy measure of A. Then*

$$\mathbf{P}(-g_0 \in dt) = \mathbf{P}(d_0 \in dt) = \frac{n^+(t,\infty)}{M}\, dt; \tag{3.3}$$

$$\mathbf{P}(V \in dv) = \frac{v}{M}\, n^+(dv), \ \text{where } V := d_0 - g_0; \tag{3.4}$$

$$\mathbf{P}(d_0 > v, -g_0 > w) = \frac{1}{M}\int_{v+w}^\infty n^+(t,\infty)\, dt; \tag{3.5}$$

$$\mathbf{P}(d_0 \in dt, -g_0 \in ds)/dt\, ds = -\frac{1}{M}\frac{d}{dt} n^+(t+s,\infty). \tag{3.6}$$

Proof: Formula (3.3) is a consequence of (3.1) and Proposition 3.1, and is given in [23] and [24, 25] (the *global formula*). Formulae (3.4), (3.5), and (3.6) follow immediately from (3.3) and Proposition 2.2. □

Remark 3.3. It is interesting to compare the right-hand sides of (3.6) and (2.5). For this, notice first from (3.2) that

$$n^+(t,\infty) = \frac{d}{dS(x)}\mathbf{P}_x(H_0 \geq t)\Big|_{x=0+}. \tag{3.7}$$

Thus, we have

$$\begin{aligned} -\frac{d}{dt}n^+(t+s,\infty) &= \frac{d}{dt}\frac{d}{dS(x)}\mathbf{P}_x(H_0 < t+s)\Big|_{x=0+} \\ &= \frac{d}{dS(x)}\frac{d}{dt}\mathbf{P}_x(H_0 < t+s)\Big|_{x=0+} \\ &= \frac{d}{dS(x)}\frac{d}{dS(y)}\widehat{p}(t+s;x,y)\Big|_{x,y=0+}. \end{aligned}$$

3.2 Spectral representations for d_0 and $V = d_0 - g_0$

In this section we show that the distribution of d_0 (and $-g_0$) is a mixture of exponential distributions and the distribution of $V = d_0 - g_0$ is a mixture of gamma distributions. The mixing measures are the same and closely related to the so called principal spectral measure of the process X, as defined in Krein's theory of strings, see [14, 16, 18]. Our starting point is the result due to Knight (see [15]) which states that the Lévy measure $n^+(dt)$ is absolutely continuous with respect to the Lebesgue measure and there exists a unique measure Δ such that

$$\nu(t) := n^+(dt)/dt = \int_0^\infty e^{-zt}\,\Delta(dz). \tag{3.8}$$

Moreover, Δ has the properties

$$\int_0^\infty \frac{\Delta(dz)}{z(z+1)} < \infty \tag{3.9}$$

and

$$\int_0^\infty \frac{\Delta(dz)}{z} = \infty. \tag{3.10}$$

We remark (cf. [15]) that (3.9) is equivalent with the defining property of the Lévy measure of a subordinator, i.e.,

$$\int_0^\infty (1 \wedge t)\, n^+(dt) < \infty.$$

The property in (3.10) is a consequence of the fact that the speed measure is strictly positive everywhere and, in particular, in a right neighbourhood of 0, see [14] p. 82. For another proof of (3.10) see [19], where Δ is interpreted as the principal spectral measure of a killed string.

Proposition 3.4. *Let Δ be the measure introduced above and associated with the process X. Then the measure*

$$\widetilde{\Delta}(dz) = \Delta(dz)/(Mz^2)$$

is a probability measure. Moreover,

$$\mathbf{P}(d_0 \in dt)/dt = \int_0^\infty z e^{-zt}\, \widetilde{\Delta}(dz), \tag{3.11}$$

and

$$\mathbf{P}(V \in dv)/dv = \int_0^\infty z^2 v\, e^{-zv}\, \widetilde{\Delta}(dz). \tag{3.12}$$

Proof: Recall that in the recurrent case (see [28] p. 220 and [6] p. 20),

$$\lim_{\alpha \searrow 0} \alpha G_\alpha(x,x) = \frac{1}{m\{I\}}, \quad \text{for all } x \in I. \tag{3.13}$$

By (3.13), (3.8), and Fubini's theorem,

$$\begin{aligned} M := m\{I\} &= \lim_{\alpha \searrow 0} \frac{1}{\alpha G_\alpha(0,0)} \\ &= \int_0^\infty n^+(t,\infty)\, dt \\ &= \int_0^\infty dt \int_t^\infty \nu(s)\, ds \\ &= \int_0^\infty dt \int_0^\infty \Delta(dz) \frac{e^{-zt}}{z} \\ &= \int_0^\infty \frac{\Delta(dz)}{z^2}, \end{aligned}$$

and, therefore, $\widetilde{\Delta}$ is a probability measure. Formulae (3.11) and (3.12) follow now from Proposition 3.2 and spectral representation (3.8). □

Remark 3.5. From the proof of Proposition 3.4 a new test for positive recurrence emerges: a recurrent diffusion X is positively recurrent if and only if

$$\int_0^\infty \frac{\Delta(dz)}{z^2} < \infty.$$

3.3 Excursion bridges

In this section we use the theory of the Palm measures to show that the result in Theorem 2.5 has a counterpart in the framework of excursions from 0 to 0 of the stationary non-negative diffusion X. Let

$$\mathcal{M} := \{t \in \mathbb{R} : X_t = 0\}$$

be the zero set of X and

$$\mathcal{L} := \{t \in \mathcal{M} \,:\, \exists\, \varepsilon > 0\ \forall\ 0 < s < \varepsilon \quad X_{t+s} > 0\}$$

be the set of the starting times of excursions from 0 to 0 on a path of X.

We introduce next the space E consisting of continuous functions $e : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $e(0) = 0$ and for which there exists $\zeta = \zeta(e) > 0$ with the property $e(t) > 0$ for $0 < t < \zeta$ and $e(t) = 0$ for $t \geq \zeta$. The space $(E, \mathcal{E})$, where $\mathcal{E}$ is the σ-algebra generated by the cylinder sets in the usual way, is called the canonical excursion space for excursions from 0 to 0 (associated with X). For $t \in \mathcal{L}$ define $X^{(ex,t)} = \{X_s^{(ex,t)} \,:\, s \geq 0\}$ where

$$X_s^{(ex,t)} := \begin{cases} X_{t+s}, & \text{for } t+s < R \\ 0, & \text{for } t+s \geq R, \end{cases}$$

and $R := \inf\{u > t \,:\, X_u = 0\}$. Clearly, $X^{(ex,t)} \in E$ and $\zeta(X^{(ex,t)}) = R$.

We follow now [23] and define the concept of equilibrium excursion measure which is a particular case of the Palm measure construction. See [23] for general results on Palm measures and further references. For background on Palm measures, especially in queueing, see, e.g., [1].

Definition 3.6. For $B \in \mathcal{E}$ let

$$\mathbf{Q}(B) := \mathbf{E}\left(\left|\{t : 0 < t < 1, t \in \mathcal{L}, X^{(ex,t)} \in B\}\right|\right), \tag{3.14}$$

where $|\cdot|$ denotes the number of elements in the set under the consideration. The measure $\mathbf{Q}$ is called the equilibrium excursion measure for the excursions of X from 0 to 0.

The next proposition and its corollary are adopted from [23] (Theorem p. 290) where a more general statement originating from [21] and [22] is given. Because of the key importance of this result for our application, we restate and formulate it especially for excursions (cf. Section III in [23] where connections with the Maisonneuve formula are discussed).

Proposition 3.7. *The measure* $\mathbf{Q}$ *is* σ*-finite and for all measurable* $f : \mathbb{R} \times E \to [0, \infty)$*,*

$$\mathbf{E}\Big(\sum_{t \in \mathcal{L}} f(t, X^{(ex,t)})\Big) = \int_{\mathbb{R}} \int_{E} ds\, \mathbf{Q}(de)\, f(s, e). \tag{3.15}$$

Proof: For the proof of the first claim, see [23]. To prove formula (3.15), observe from the definition of $\mathbf{Q}$, that (3.15) is equivalent with

$$\mathbf{E}\Big(\sum_{t \in \mathcal{L}} f(t, X^{(ex,t)})\Big) = \mathbf{E}\Big(\sum_{t \in \mathcal{L} \cap (0,1)} \int_{\mathbb{R}} ds f(s, X^{(ex,t)})\Big). \tag{3.16}$$

Let $\theta = \{\theta_t : t \in \mathbb{R}\}$ be the usual shift operator defined in the underlying probability space via

$$X_s \circ \theta_t(\omega) = X_{t+s}(\omega) \quad \forall t, s \in \mathbb{R}.$$

Notice that the law of X is invariant under θ; this is, in our case, equivalent with the stationarity of X. Consequently, by stationarity,

$$\begin{aligned}
&\mathbf{E}\Big(\sum_{t \in \mathcal{L} \cap (0,1)} \int_{\mathbb{R}} ds\, f(s, X^{(ex,t)})\Big) \\
&\qquad = \int_{\mathbb{R}} ds\ \mathbf{E}\Big(\sum_{t \in \mathcal{L} \circ \theta_s \cap (0,1)} f(s, X^{(ex,t)} \circ \theta_s)\Big) \\
&\qquad = \int_{\mathbb{R}} ds\ \mathbf{E}\Big(\sum_{t \in (0,1), t+s \in \mathcal{L}} f(s, X^{(ex,t+s)})\Big) \\
&\qquad = \int_0^1 dt\ \mathbf{E}\Big(\sum_{s \in \mathbb{R}, s \in \mathcal{L} \circ \theta_t} f(s, X^{(ex,s)} \circ \theta_t)\Big) \\
&\qquad = \int_0^1 dt\ \mathbf{E}\Big(\sum_{s \in \mathcal{L}} f(s, X^{(ex,s)})\Big) \\
&\qquad = \mathbf{E}\Big(\sum_{s \in \mathcal{L}} f(s, X^{(ex,s)})\Big),
\end{aligned}$$

hence (3.16) is proved giving (3.15). □

Corollary 3.8. *For a jointly measurable* $h : \mathbb{R} \times E \mapsto [0, \infty)$

$$\mathbf{E}\left(h(-g_0, X^{(ex,g_0)})\right) = \int_E \mathbf{Q}(de)\Big(\int_0^{\zeta(e)} h(s,e)\,ds\Big). \tag{3.17}$$

In particular, for $a > 0,\ v > 0,$ *and* $e \in E$

$$\mathbf{P}(-g_0 \in da,\ X^{(ex,g_0)} \in de) = da\,\mathbf{Q}(de)\,\mathbf{1}_{(a<\zeta(e))}, \tag{3.18}$$

$$\mathbf{P}(X^{(ex,g_0)} \in de) = \mathbf{Q}(de)\zeta(e), \tag{3.19}$$

$$\mathbf{P}(-g_0 \in da)/da = \mathbf{Q}(\zeta > a), \tag{3.20}$$

and

$$\mathbf{P}(V \in dv) = v\,\mathbf{Q}(\zeta(e) \in dv). \tag{3.21}$$

Proof: To prove formula (3.17), we set in (3.15)

$$f(t,e) = h(-t,e)\mathbf{1}_{(0<-t<\zeta(e))}$$

with h jointly measurable (cf. [23] p. 291 and [22] p. 332). Then, due to the particular form of f,

$$\begin{aligned}\mathbf{E}\Big(\sum_{t\in\mathcal{L}} f(t, X^{(ex,t)})\Big) &= \mathbf{E}\ \left(h(-g_0, X^{(ex,g_0)})\right) \\ &= \int_E \mathbf{Q}(de)\Big(\int_0^{\zeta(e)} h(s,e)\,ds\Big).\end{aligned} \tag{3.22}$$

Choosing h appropriately in (3.17) yields (3.18), (3.19), and (3.20). Finally, take in (3.22) $h(s,e) = \mathbf{1}_B(e),\ B = \{\zeta > v\},\ v > 0$, to obtain

$$\mathbf{P}\,(V > v) = \mathbf{Q}\,(\zeta(e)\,;\,\zeta(e) > v)$$

which is equivalent with (3.21). □

Remark 3.9. 1) From (3.18) and (3.19) it follows that the distribution of $-g_0$ given the excursion is uniform on $(0, V)$ and, in particular, $(-g_0, d_0) \in \mathcal{K}$ (cf. Proposition 2.3).
2) It is seen from Proposition 3.2 and Corollary 3.8 that

$$\mathbf{Q}(\zeta(e) \in dv) = \frac{n^+(dv)}{M}.$$

To proceed, let $\mathbf{M}$ denote the Itô excursion law for the excursions of X from 0 to 0. Then it is well-known that

$$\mathbf{M}(\zeta > a) = c \int_0^\infty m(dx)\, n_x(0, a),$$

where c is a normalizing constant and

$$n_x(0, a) := \mathbf{P}(H_0 \in da)/da$$

is the density of the first hitting time $H_0 := \inf\{t : X_t = 0\}$ given that $X_0 = x$. Now (3.20) yields

$$\mathbf{M}(\zeta > a) = c'\mathbf{Q}(\zeta > a)$$

for some (normalizing) constant c'. In fact, as can be deduced from (3.18), the measures $\mathbf{M}$ and $\mathbf{Q}$ are the same (up to a constant):

Proposition 3.10. *There exists a constant c' such that for all $B \in \mathcal{E}$*

$$\mathbf{M}(B) = c'\mathbf{Q}(B). \tag{3.23}$$

We refer to [23] for a discussion about (3.23) and the associated normalizations. See also [5] for the Brownian motion case.

To formulate the main result of this section, we need the concept of excursion bridge of X. For reflecting Brownian motion the excursion bridge is a 3-dimensional Bessel bridge from 0 to 0 (of some length l). In general, the excursion bridge of X from 0 to 0 of the length l can be described as the process $X^{(0,l,0)}$ obtained from X with $X_0 = x > 0$ conditioned to hit 0 for the first time at time l by letting $x \to 0$. Clearly, defining $X_t^{(0,l,0)} = 0$ for $t \geq l$ it holds $X^{(0,l,0)} \in E$.

Theorem 3.11. *Let $X^{(0,l,0)} = \{X_t^{(0,l,0)} : t \geq 0\}$ be the excursion bridge as defined above and let U be a uniformly on $(0, l)$ distributed random variable independent of $X^{(0,l,0)}$. Let*

$$I^{(l,+)} := \int_0^l \mathbf{1}_{\{X_s^{(0,l,0)} > X_U^{(0,l,0)}\}}\, ds$$

and

$$I^{(l,-)} := \int_0^l \mathbf{1}_{\{X_s^{(0,l,0)} < X_U^{(0,l,0)}\}}\, ds.$$

Then $I^{(l,+)}$ and $I^{(l,-)}$ are identically and uniformly on $(0, l)$ distributed random variables.

Proof: We use formula (3.17) and let therein

$$h(s,e) = F(\zeta(e))\, G\Big(\int_0^{\zeta(e)} \mathbf{1}_{\{e_t > e_s\}} dt\Big)\, \mathbf{1}_{\{s<\zeta(e)\}}$$

with Borel-measurable functions F and G. Recall the notation $V = d_0 - g_0$, and consider first the left-hand side of (3.17) with h as above:

$$\begin{aligned}
\mathbf{E}\left(h(-g_0, X^{(ex,g_0)})\right) &= \mathbf{E}\Big(F(V)\, G\Big(\int_0^V \mathbf{1}_{\{X_t^{(ex,g_0)} > X_{-g_0}^{(ex,g_0)}\}} dt\Big)\, \mathbf{1}_{\{-g_0<V\}}\Big) \\
&= \mathbf{E}\Big(F(V) G\Big(\int_0^V \mathbf{1}_{\{X_{g_0+t} > X_0\}} dt\Big)\Big) \\
&= \mathbf{E}\left(F(V)\, G(I^{(+)})\right), \qquad (3.24)
\end{aligned}$$

where, as before,

$$I^{(+)} := \int_{g_0}^{d_0} \mathbf{1}_{\{X_t > X_0\}} dt.$$

For the right-hand side of (3.17) we have

$$\begin{aligned}
&\int_E \mathbf{Q}(de)\Big(\int_0^{\zeta(e)} h(s,e)\, ds\Big) \\
&\quad = \mathbf{Q}\Big(F(\zeta(e))\, \zeta(e) \int_0^{\zeta(e)} G\Big(\int_0^{\zeta(e)} \mathbf{1}_{\{e_t>e_s\}} dt\Big)\, \frac{ds}{\zeta(e)}\Big) \\
&\quad = \frac{1}{c'}\, \mathbf{M}\Big(F(\zeta(e))\, \zeta(e) \int_0^{\zeta(e)} G\Big(\int_0^{\zeta(e)} \mathbf{1}_{\{e_t>e_s\}} dt\Big)\, \frac{ds}{\zeta(e)}\Big), \qquad (3.25)
\end{aligned}$$

where in the last step Proposition 3.10 is used. Next recall that the description of the Itô measure via the lengths of the excursions (see, e.g., [26] p. 497, [6] p. 60, [27] p. 421) says that for all $B \in \mathcal{E}$,

$$\mathbf{M}(B) = \int_0^\infty \mathbf{M}(\zeta \in dl)\, \mathbf{P}^{(0,l,0)}(B),$$

where $\mathbf{P}^{(0,l,0)}$ is the probability measure associated with the excursion bridge process $X^{(0,l,0)}$ defined in the canonical excursion space E. Consequently, we obtain

$$
\begin{aligned}
&\mathbf{M}\Big(F(\zeta(e))\,\zeta(e)\int_0^{\zeta(e)} G\Big(\int_0^{\zeta(e)} \mathbf{1}_{\{e_t>e_s\}}dt\Big)\,\frac{ds}{\zeta(e)}\Big)\\
&\quad = \int_0^\infty \mathbf{M}(\zeta(e)\in dl)\,F(l)\,l\,\mathbf{E}^{(0,l,0)}\Big(\int_0^l G\Big(\int_0^l \mathbf{1}_{\{e_t>e_s\}}dt\Big)\,\frac{ds}{l}\Big)\\
&\quad = \int_0^\infty \mathbf{M}(\zeta(e)\in dl)\,F(l)\,l\,\mathbf{E}^{(0,l,0)}\Big(G\Big(\int_0^l \mathbf{1}_{\{e_t>e_U\}}dt\Big)\Big), \qquad (3.26)
\end{aligned}
$$

where U is uniformly on $(0,l)$ distributed random variable independent of $X^{(0,l,0)}$. By (3.17), the right-hand sides of (3.24) and (3.25) are equal. Consequently, using (3.21) in Corollary 3.8 and (3.26), we obtain

$$
\int_0^\infty \mathbf{P}(V\in dl)F(l)\,\mathbf{E}\left(G(I^{(+)})|V=l\right) = \int_0^\infty \mathbf{P}(V\in dl)F(l)\,\mathbf{E}^{(0,l,0)}\left(G(I^{(l,+)})\right)
$$

giving

$$
\mathbf{E}\left(G(I^{(+)})|V=l\right) = \mathbf{E}^{(0,l,0)}\left(G(I^{(l,+)})\right).
$$

Combining this with the result in Theorem 2.5 completes the proof. □

4 Remarks on the null recurrent case

Remark 4.1. Suppose now that X is null recurrent. Then the speed measure $m(dx)$ serves still as the stationary measure of X but because $m\{I\} = \infty$ it cannot be normalized to a probability measure. However, the result in Theorem 2.5 is also valid in this case:

$$
\begin{aligned}
\int_I m(dx)\,\mathbf{E}_x\big(e^{-\alpha I^{(+)}-\beta I^{(-)}}\big) &= \int_I m(dx)\,\mathbf{E}_x\big(e^{-\alpha d_0+\beta g_0}\big)\\
&= \frac{1}{\alpha-\beta}\left(\frac{1}{G_\alpha(0,0)}-\frac{1}{G_\beta(0,0)}\right). \qquad (4.1)
\end{aligned}
$$

In particular, given that $V := d_0 - g_0 = v$, $I^{(+)}$ and $I^{(-)}$ are uniformly distributed on $(0,v)$. In the case when X is a stationary reflecting Brownian motion living above zero (4.1) gives

$$\begin{aligned}\int_I m(dx)\,\mathbf{E}_x\big(e^{-\alpha I^{(+)}-\beta I^{(-)}}\big) &= \int_I m(dx)\,\mathbf{E}_x\big(e^{-\alpha d_0+\beta g_0}\big)\\ &= \frac{1}{\alpha-\beta}\left(\sqrt{2\alpha}-\sqrt{2\beta}\right)\\ &= \frac{\sqrt{2}}{\sqrt{\alpha}+\sqrt{\beta}}.\end{aligned}$$

It is also easily checked that finiteness of m is not needed in the proof of Theorem 3.11 and, hence, for all diffusion excursion bridges $X^{(0,l,0)}$ of length l the random variables $I^{(l,+)}$ and $I^{(l,-)}$ (defined as in Theorem 3.11) are uniformly distributed on $(0,l)$.

Remark 4.2. For a Brownian excursion there is an alternative simple proof of the result in Theorem 3.11 based on Vervaat's path transformation (see [33, 4, 2]). Indeed, let $X^{(0,l,0)}$ denote a standard Brownian excursion of length l, that is a 3-dimensional Bessel bridge of length l, and U be a random variable uniformly distributed on $(0,l)$ and independent of $X^{(0,l,0)}$. Then the process $X^{br}=\{X_t^{br}:0\le t\le l\}$ defined as

$$X_t^{br}=\begin{cases} X_{t+U}^{(0,l,0)}-X_U^{(0,l,0)}, & t+U\le l,\\ X_{t+U-l}^{(0,l,0)}-X_U^{(0,l,0)}, & t+U\ge l\end{cases}$$

equals in distribution to a standard Brownian bridge of length l. Clearly, with the notation in Theorem 3.11,

$$I^{(l,+)}=\int_0^l \mathbf{1}_{\{X_s^{br}>0\}}\,ds=:I^{(br,+)}$$

and

$$I^{(l,-)}=\int_0^l \mathbf{1}_{\{X_s^{br}<0\}}\,ds=:I^{(br,-)}.$$

The claim follows now from the well-known result due to Lévy, saying that $I^{(br,+)}$ and $I^{(br,-)}$ are uniformly distributed on $(0,l)$ (see, e.g., [6] p. 163 and [35] p. 43).

5 Examples

To illustrate the results in Section 2 we consider some examples of stationary diffusions in stationary state.

Example 5.1. (Reflecting Brownian motion with drift) Let X be a stationary reflecting Brownian motion with drift $-\mu < 0$ living above zero. The speed measure is

$$m(dx) = 2e^{-2\mu x}\,dx,$$

the scale function is

$$S(x) = \frac{1}{2\mu}(1 - e^{2\mu x}),$$

and the Green function at $(0,0)$ is

$$G_\alpha(0,0) = \frac{1}{\sqrt{2\alpha + \mu^2} - \mu}.$$

Hence by Theorem 2.5 and (2.4) (cf. [29]),

$$\begin{aligned}
\mathbf{E}\big(e^{-\alpha I^{(+)} - \beta I^{(-)}}\big) &= \mathbf{E}\left(e^{-\alpha d_0 + \beta g_0}\right) \\
&= \frac{\mu}{\alpha - \beta}\Big((\sqrt{2\alpha + \mu^2} - \mu) - (\sqrt{2\beta + \mu^2} - \mu)\Big) \\
&= \frac{2\mu}{\sqrt{2\alpha + \mu^2} + \sqrt{2\beta + \mu^2}}. \qquad (5.1)
\end{aligned}$$

The transition density for the killed process is

$$\widehat{p}(t;x,y) = \frac{1}{2\sqrt{2\pi t}} e^{\mu(x+y) - \frac{\mu^2 t}{2}} \left(e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}}\right).$$

Thus by (2.5), the joint density of $(-g_0, d_0)$ and $(I^{(+)}, I^{(-)})$ is

$$\begin{aligned}
f(t,s) &= \mu\left(\frac{d}{dx}\frac{d}{dy}\widehat{p}(s+t;x,y)/(S'(x)S'(y))\right)\Big|_{x,y=0} \\
&= \frac{\mu}{\sqrt{2\pi(t+s)^3}} e^{-\frac{\mu^2}{2}(t+s)}.
\end{aligned}$$

To check the obtained formulae we give here also the following expressions for the corresponding Laplace transforms when X started at x:

$$\mathbf{E}_x\left(\exp\Big(-\alpha\int_0^{H_0(X)}\mathbf{1}_{\{0\le X_s<x\}}\,ds-\beta\int_0^{H_0(X)}\mathbf{1}_{\{X_s>x\}}\,ds\Big)\right)$$
$$\frac{\sqrt{2\alpha+\mu^2}e^{\mu x}}{\sqrt{2\alpha+\mu^2}\cosh(x\sqrt{2\alpha+\mu^2})+\sqrt{2\beta+\mu^2}\sinh(x\sqrt{2\alpha+\mu^2})}$$

(see [6], formula 2.2.6.1 p. 300), while

$$\mathbf{E}_x\left(e^{-\alpha H_0(X)}\right)=e^{(\mu-\sqrt{2\alpha+\mu^2})x}.$$

Example 5.2. (Brownian motion reflected at 0 **and** 1**)** Consider now a stationary Brownian motion X living in the interval $I=[0,1]$ and reflected at 0 and at 1. The speed measure of X is $m(dx)=2\,dx$, the scale function $S(x)=x$, and the Green function at $(0,0)$ is (see [6], p. 122)

$$G_\alpha(0,0)=\frac{\coth(\sqrt{2\alpha})}{\sqrt{2\alpha}}.$$

Hence the Laplace transform of $(-g_0,d_0)$ and $(I^{(+)},I^{(-)})$ is $(\alpha\neq\beta)$

$$\begin{aligned}\mathbf{E}\big(e^{-\alpha I^{(+)}-\beta I^{(-)}}\big)&=\mathbf{E}\big(e^{-\alpha d_0+\beta g_0}\big)\\&=\frac{1}{\alpha-\beta}\Big(\sqrt{\alpha/2}\tanh(\sqrt{2\alpha})-\sqrt{\beta/2}\tanh(\sqrt{2\beta})\Big)\end{aligned}$$

and

$$\mathbf{E}(e^{-\alpha(d_0-g_0)})=\frac{1}{4\sqrt{2\alpha}\cosh^2(\sqrt{2\alpha})}\Big(\sinh(2\sqrt{2\alpha})+2\Big).$$

Example 5.3. (Squared radial Ornstein-Uhlenbeck process) Let $X=\{X_t:t\in\mathbb{R}\}$ be a stationary squared radial Ornstein-Uhlenbeck process with parameters $\nu=n/2-1$ and γ. The generator of $\{X_t:t\ge 0\}$ is

$$\mathcal{G}=2x\,\frac{d^2}{dx^2}+(n-2\gamma\,x)\frac{d}{dx}.$$

The process X can be defined as

$$X(t)=e^{-\gamma t}SQBES(e^{2\gamma t}/2\gamma),\quad t\in\mathbb{R}, \tag{5.2}$$

where $SQBES$ is a squared Bessel process of dimension $n = 2\nu+2$ (see [24]). Assume that $-1 < \nu < 0$ and 0 is a reflecting boundary. Then the process is positively recurrent with the speed measure given by (see [6] p. 140)

$$m(dx) = \frac{1}{2} x^{\nu} e^{-\gamma x}\, dx,$$

that is (after normalization) the gamma-density with parameters γ and $\nu+1$. The Green function at $(0,0)$ is given by (see [6] p. 141)

$$G_{\alpha}(0,0) = \gamma^{\nu} \frac{\mathrm{B}(\frac{\alpha}{2\gamma}, -\nu)}{\Gamma(1+\nu)},$$

where $\Gamma(x)$ and $\mathrm{B}(x,y)$ denote the gamma and beta functions, respectively. By Theorem 2.5,

$$\begin{aligned}\mathbf{E}\big(e^{-\alpha I^{(+)} - \beta I^{(-)}}\big) = \mathbf{E}\left(e^{-\alpha d_0 + \beta g_0}\right) &= \frac{1}{M(\alpha-\beta)}\Big(\frac{1}{G_{\alpha}(0,0)} - \frac{1}{G_{\beta}(0,0)}\Big) \\ &= \frac{2\gamma}{\alpha-\beta}\Big(\frac{1}{\mathrm{B}(\frac{\alpha}{2\gamma}, -\nu)} - \frac{1}{\mathrm{B}(\frac{\beta}{2\gamma}, -\nu)}\Big).\end{aligned}$$

Letting $\beta = 0$ and taking the inverse Laplace transform (see [11] p. 261) gives the density of d_0 (and $I^{(\pm)}$) and the expression for $n^{+}(t,\infty)/M$:

$$\mathbf{P}(d_0 \in dt)/dt = \frac{n^{+}(t,\infty)}{M} = \frac{2\gamma}{\Gamma(-\nu)\Gamma(1+\nu)} e^{2\mu\nu t}\left(1 - e^{-2\mu t}\right)^{\nu}. \qquad (5.3)$$

Formula (5.3) can be also obtained using (5.2) as in [24]. The joint density of d_0 and g_0 (and $I^{(\pm)}$) is given then by

$$f(t,s) = -\frac{1}{M}\frac{d}{dt} n^{+}(t+s,\infty) = -\frac{4\mu\nu\gamma}{\Gamma(-\nu)\Gamma(1+\nu)} e^{2\mu\nu(s+t)}\left(1 - e^{-2\mu(s+t)}\right)^{\nu-1}.$$

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