A storage process with local time input

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Abstract

In this paper we introduce a storage process with singular continuous input. The input process is defined as the local time of a stationary reflecting Brownian motion with drift. Many basic charateristics of the process are computed explicitly, e.g., stationary distribution, distributions of the starting and ending time of on-going busy and idle periods. We also consider the multifractal spectrum of the input process and observe that it is independent of system parameters.

1 Introduction

Storage models with continuous input have been thoroughly studied with piecewise differentiable input (starting from Kosten [15]) and with non-differentiable input like diffusions (see, e.g., [8]) and fractional Brownian motion (see, e.g., [16]). To our knowledge, no exact results have so far been obtained on storages with continuous singular input. In this paper we consider a model where the input is the local time process of a diffusion, more precisely, the local time at zero of a reflecting Brownian motion with negative drift $-\mu$. This is a continuous increasing process whose derivative is zero almost everywhere. We derive the stationary distribution of the storage occupancy (queue length) and the characteristics of its busy and idle periods.

The process considered in this paper could be used to model systems like water reservoirs — at least as a toy model. In our approach the input comes in

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infinitesimal droplets, whereas usually (see, e.g., Prabhu [18]) dam models are constructed using Lévy processes.

It is rather interesting that although the input process is exotic enough to have a non-trivial multifractal spectrum, the queue length distribution is just the exponential distribution plus an atom at the origin. Moreover, the spectrum does not even depend on the system parameter μ. This shows that although multifractal scaling properties of telecommunication traffic have recently gained interest (see, e.g., [17]), the multifractal spectrum as such gives hardly any information on system performance.

The paper is structured as follows. The system is defined in section 2 and a real, although somewhat fictitious queueing system, for which our system may be obtained as a limit, is described in section 3. Section 4 makes some observations on the point process related to the alternating busy and idle periods. The stationary queue length distribution is found in section 5, after which the distribution of the length of the on-going idle period is computed in section 6. In section 7, we derive the common distribution of the starting and ending time of the on-going busy period and notice, surprisingly, that this is the same as the corresponding distribution for the on-going idle period except that μ is replaced by $1-\mu$. Finally, the multifractal spectrum of the input process is computed in section 8.

2 System definition

In this section, we introduce the storage process with local time input, see Definition 2.3. Before to be able to do that we must construct the needed local time process, which should be defined on the whole time axis $\mathbb{R}$ and have stationary increments. It is seen that we may use local times of two (conditionally) independent copies of a linear diffusion in stationary state. In fact, we take the underlying diffusion to be a stationary reflecting Brownian motion with drift $-\mu < 0$, RBM($-\mu$), for short. For RBM($-\mu$) the local time can also be constructed using Skorokhod's reflection lemma, and this approach is also practical for our purposes. To study RBM($-\mu$) in details is motivated because RBM($-\mu$) is a much used storage process due to the fact that it is obtained as a heavy traffic limit of many discrete queueing models. We recall below some properties of RBM($-\mu$) needed here, and refer to Harrison [8], Abate and Whitt [1],[2], and Salminen and Norros [21] for further readings.

We let $X = \{X_t \ : \ t \in \mathbb{R}\}$ be a reflecting Brownian motion in stationary state living on $[0,+\infty)$ with drift $-\mu < 0$. The stationary distribution of X is an exponential distribution with parameter 2μ, and we let $m(dx) = 2\mu\exp(-2\mu x)\,dx$ denote the corresponding measure. It is well known that the transition probability is absolutely continuous with respect to m, and we have

$$\mathbf{P}(X_t \in dy \mid X_s = x) = \tilde{p}(t-s;x,y)\,m(dy),$$

where the symmetric (in x and y) transition density is given by (see Harrison [8] p. 49 for the distribution function, and Borodin and Salminen [7] p. 129 for the density)

$$\tilde{p}(t;x,y) = \frac{1}{2\mu\sqrt{2\pi t}}\,\mathrm{e}^{\mu(y+x)-\frac{\mu^2 t}{2}}\left(\mathrm{e}^{-\frac{(x-y)^2}{2t}} + \mathrm{e}^{-\frac{(x+y)^2}{2t}}\right) + \Phi\left(\frac{-y-x+\mu t}{\sqrt{t}}\right)$$

and Φ is the standard normal distribution function. For any $t_1 < t_2 < \ldots < t_n$ and Borel sets $A_1, A_2, \ldots, A_n$, the finite dimensional distributions of X are given by

$$\begin{aligned}&\mathbf{P}(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n)\\ &= \int_{A_1} m(dx_1)\ldots\int_{A_n} m(dx_n)\,\tilde{p}(t_2-t_1;x_1,x_2)\ldots\tilde{p}(t_n-t_{n-1};x_{n-1},x_n).\end{aligned}$$

A key property of X needed hereby is that X is reversible in time, i.e.,

$$\{X_t : t \in \mathbb{R}\} \quad \sim \quad \{X_{-t} : t \in \mathbb{R}\}. \tag{1}$$

Let now $X^{(1)}$ and $X^{(2)}$ be two copies of $\{X_t : t \geq 0\}$. Assume that $X_0^{(1)} = X_0^{(2)}$ but let $X^{(1)}$ and $X^{(2)}$ be otherwise independent. From (1) it is seen that $\{X_t : t \in \mathbb{R}\}$ is identical in law to $\{Z_t : t \in \mathbb{R}\}$ where

$$Z_t := \begin{cases} X_{-t}^{(1)}, & t \leq 0,\\ X_t^{(2)}, & t \geq 0.\end{cases}$$

Introduce the local times of $X^{(1)}$ and $X^{(2)}$ at 0 via

$$L_t^{(i)} := \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t \mathbf{1}_{[0,\varepsilon)}(X_s^{(i)})\,ds,\ i = 1,2, \tag{2}$$

respectively, where the limits exist almost surely. Using $L^{(1)}$ and $L^{(2)}$ we construct the local time process $L = \{L_t : t \in \mathbb{R}\}$, playing the main role in this paper, by setting

$$L_t := \begin{cases} -L_{-t}^{(1)}, & t \leq 0,\\ L_t^{(2)}, & t \geq 0.\end{cases} \tag{3}$$

Notice that $L_0 = 0$ by construction and $t \mapsto L_t$ is continuous and non-decreasing. A crucial property of L (stated also in Proposition 2.2) is that it has stationary increments. Recall

Definition 2.1 *A stochastic process $Y = \{Y_t \ : \ t \in \mathbb{R}\}$ is said to have stationary increments, if*

$$(Y_{s+t} - Y_s)_{t\in\mathbb{R}} \stackrel{\mathrm{d}}{=} (Y_t - Y_0)_{t\in\mathbb{R}}$$

for all $s \in \mathbb{R}$, where $\stackrel{\mathrm{d}}{=}$ means equality in distribution.

To prove Proposition 2.2 and also for developments in Section 3, we consider an alternative construction of L via Skorokhod's reflection lemma (see Harrison [8] p. 18, Revuz and Yor [22] p. 239). For this, let $W = \{W_t \ : \ t \in \mathbb{R}\}, W_0 = 0$, be a standard Brownian motion defined on the whole time axis $\mathbb{R}$ (see [21]). In particular, W is a Gaussian process with stationary and independent increments. Further, let $\mu > 0$ and denote by

$$W_t^{(\mu)} \doteq W_t - \mu\, t$$

the corresponding Brownian motion with drift $-\mu$. The stationary reflecting Brownian motion with drift $-\mu < 0$ can be interpreted as a storage process, where the cumulative input is a standard Brownian motion, and the storage is emptied with rate μ, that is, in this framework we define

$$X_t \doteq \sup_{s\le t}(W_t^{(\mu)} - W_s^{(\mu)}), \quad t \in \mathbb{R}. \tag{4}$$

By the construction, X is a non-negative stationary process. It follows from Skorokhod's lemma when considering the equality

$$X_t = K_0 + W_t^{(\mu)} + L_t, \quad t \in \mathbb{R}.$$

as an equation that the local time of X is given by

$$L_t = K_t - K_0, \quad t \in \mathbb{R}. \tag{5}$$

where $K_t \doteq -\inf_{s\le t} W_s^{(\mu)}$. From this it is apparent that $L_0 = 0$ and $t \mapsto L_t$ is continuous and non-decreasing.

Proposition 2.2 *The process L has stationary increments, and*

$$\mathbf{E}L_t = \mu t. \tag{6}$$

Proof Observe first that if a process $(Y_t)_{t\in\mathbb{R}}$ has stationary increments, then the process $Y_t^* = \sup_{s\le t} Y_s$ also has stationary increments. Consequently, from (5), L has stationary increments. Since L_1 is clearly integrable, it follows that $\mathbf{E}L_t = t\,\mathbf{E}L_1$. Finally, the stationarity of X implies (since $X_0 = K_0$)

$$\mathbf{E}L_1 = \mathbf{E}X_1 - \mathbf{E}K_0 - \mathbf{E}W_1^{(\mu)} = \mu.$$

□

Using the process L we define now the basic object of our study.

Definition 2.3 *The process $S = \{S_t : t \in \mathbb{R}\}$ defined via*

$$S_t := \sup_{s\le t}\{L_t - L_s - (t-s)\}, \quad t \in \mathbb{R}.$$

is called a storage process with local time input, service rate 1, and unbounded buffer, associated to the reflecting Brownian motion with drift $-\mu$.

Remark 2.4 The key property when construction storage processes with constant service rate is that the input process has stationary increments. It can be proved that the local time defined as in (2) and (3) but for an arbitrary linear stationary diffusion (in stationary state) has stationary increments. This proof (which we do not present here) cannot, of course, be based on Skorokhod's reflection lemma but utilizes symmetry properties of linear diffusions. Hence, it is possible to construct a storage process as in Definition 2.3 using local time at zero of some other stationary linear diffusion. Many of our results below can be formulated for storage processes where the RBM is replaced by some other diffusion. These generalizations are treated in a forthcoming paper.

3 Motivation by a limit procedure

Our local time input process can be considered as a continuous analog to some on/off processes. We show in this section how it models the output of the lower priority class queue in a priority system where the higher priority class approaches a heavy traffic limit.

Consider, for example, a two-class pre-emptive priority system where the high priority class is an $M/M/1$ system. Let its service times be Exp(1) distributed and the arrival rate of the n'th system be

$$\lambda_n = 1 - \frac{2\mu}{\sqrt{n}}.$$

For simplicity, we do not care about stationarity here and start with an empty system at time $t = 0$.

Let $C_t^{(n)}$ be the compound Poisson process of arriving work. Then

$$\mathrm{Var}(C_t^{(n)}) = \lambda_n \mathbf{E}(\xi^2)t = 2\lambda_n t,$$

where ξ is an Exp(1) distributed random variable. Denote the cumulative idle time process of the high priority system by $I_t^{(n)}$. It can be written as (draw a picture!)

$$I_t^{(n)} = -\inf_{s\in[0,t]}(C_s^{(n)} - s).$$

Let us assume that the lower priority queue be saturated. The cumulative lower priority output process is then simply $I_t^{(n)}$.

Asymptotically, the part of time that the high priority system is empty is $1 - \lambda_n = 2\mu/\sqrt{n}$. Define the scaled process

$$L_t^{(n)} \doteq \frac{1}{\sqrt{n}} I_{nt/2}^{(n)}.$$

Note that $L_t^{(n)}$ grows asymptotically as μt. By Donsker's theorem and the continuous mapping theorem, we have

$$\begin{aligned}
L_t^{(n)} &= -\inf_{s\in[0,t/2]} \frac{C_{ns}^{(n)} - ns}{\sqrt{n}} \\
&= -\inf_{s\in[0,t/2]} \left(\sqrt{2\lambda_n} \cdot \frac{C_{ns}^{(n)} - \lambda_n ns}{\sqrt{\mathrm{Var}(C_n^{(n)})}} - 2\mu s \right) \\
&\overset{\text{(w)}}{\to} -\inf_{s\in[0,t/2]} \left(\sqrt{2}W_s - 2\mu s\right) \\
&\overset{\text{d}}{=} -\inf_{s\in[0,t]} (W_s - \mu s) \\
&\doteq L_t^0,
\end{aligned}$$

where $\overset{\text{(w)}}{\to}$ denotes weak convergence and $\overset{\text{d}}{=}$ equality in distribution.

Moreover, the marginal distributions of L_t^0 can be characterized from this limit relation. Indeed, $C_t^{(n)} - t$ is a Lévy process with Laplace exponent

$$\phi_n(\alpha) = \alpha - \lambda_n(1 - \beta(\alpha)) = \alpha\left(1 - \frac{\lambda_n}{1+\alpha}\right),$$

that is,

$$\mathbf{E}\mathrm{e}^{-\alpha(C_t^{(n)}-t)} = \mathrm{e}^{t\phi_n(\alpha)},$$

where $\beta(\alpha) = 1/(1+\alpha)$ is the Laplace transform of the exponential service time distribution. Define further the first passage time processes

$$\tau_x^{(n)} = \inf\{t > 0 : \, I_t^{(n)} = x\}.$$

It is well known (see [5]) that $\tau^{(n)}$ is a subordinator , i.e., an increasing Lévy process, with Laplace exponent $\kappa_n(\alpha) = \phi_n^{-1}(\alpha)$. Note now that

$$\tilde{\tau}_x^{(n)} \doteq \inf\{t > 0 : \, \frac{1}{\sqrt{n}} L_t^{(n)} = x\} = \frac{2}{n}\,\tau_{\sqrt{n}x}^{(n)},$$

and the Laplace exponent of the subordinator $\tilde{\tau}^{(n)}$ is

$$\tilde{\kappa}_n(\alpha) = \sqrt{n}\,\kappa_n(\frac{2\alpha}{n}).$$

A direct computation yields

$$\begin{aligned}
\kappa(\alpha) &\doteq \lim_{n\to\infty} \tilde{\kappa}_n(\alpha) \\
&= \lim_{n\to\infty} \frac{1}{2}\left(2\mu - \frac{2\alpha}{\sqrt{n}} - \sqrt{\left(2\mu - \frac{2\alpha}{\sqrt{n}}\right)^2 + 8\alpha}\right) \\
&= \mu - \sqrt{2\alpha + \mu^2}. \qquad (7)
\end{aligned}$$

By general results on weak convergence of subordinators and their inverses, it follows that $\kappa(\alpha)$ is the Laplace exponent of the right-continuous inverse of L^0, say $A_x \doteq \inf\{t > 0 : \, L_t^0 > x\}$:

$$\mathbf{E}e^{-\alpha A_x} = \mathrm{e}^{x\kappa(\alpha)}.$$

4 A point process view on the busy and idle periods

In contrast to the process X, which never remains at zero for an interval of positive length, the process S almost surely hits zero only for intervals of positive length. Indeed, because the input is singular, it can never flow through the server without

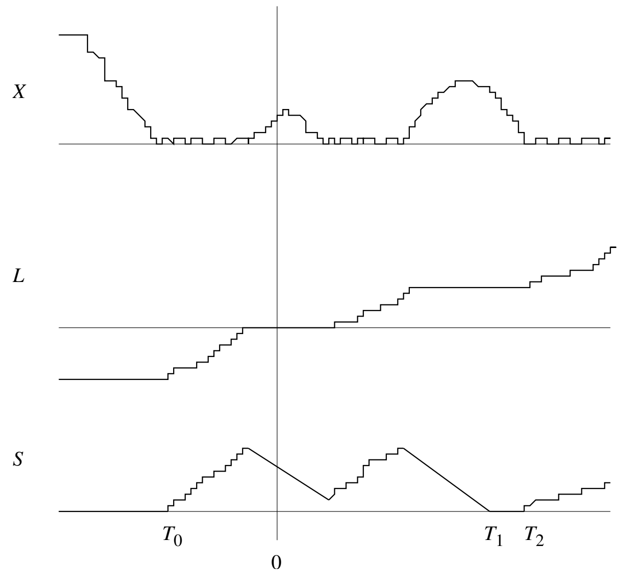

Figure 1: A schematic view of the processes X, L and S, and the time points T_0, T_1, T_2. The marks are: $Z_0 = 1$, $Z_1 = 0$, $Z_2 = 1$.

creating a positive storage. Thus the busy and idle periods of the second storage are well defined, and the fraction of time that the system is busy is $\mathbf{E}L_t/t = \mu$. Figure 1 shows the relationships between the activities of the two storages X and S.

The starting points of the busy and idle periods can be described as a stationary marked point process $(T_n)_{n\in\mathbb{Z}}$, where the mark Z_n associated with point T_n is 0 if an idle period starts at T_n and 1 if a busy period starts at T_n. For stationary point processes, we follow [3].

Let N denote the counting measure related to (T_n). The Palm probability measure $\mathbf{P}_N^0$ is defined as

$$\mathbf{P}_N^0(A) = \frac{1}{\lambda t}\mathbf{E}\int_0^t 1_A \circ \theta_s N(ds), \tag{8}$$

where $t > 0$ is arbitrary, λ is the intensity of N and θ_s is the time shift flow on Ω (see [3] for the rules on θ_s). The inverse relationship between the two probability measures is given by Ryll-Nardzewski's formula

$$\mathbf{P}(A) = \lambda \int_0^\infty \mathbf{P}_N^0(T_1 > t,\, \theta_t \in A)dt. \tag{9}$$

The well-known fact that the consumed and remaining parts of the busy period going on at time 0 are identically distributed is most elegantly proved with the Palm calculus: just compute the joint distribution of (T_0, T_1) in terms of the Palm probability:

$$
\begin{aligned}
& \mathbf{P}(T_1 > v, -T_0 > w, Z_0 = 1) \\
= \ & \lambda \int_0^\infty \mathbf{P}_N^0(T_1 > t, T_1 - t > v,\ t > w, Z_0 = 1)dt \\
= \ & \lambda \int_w^\infty \mathbf{P}_N^0(T_1 > t + v, Z_0 = 1)dt \\
= \ & \lambda \int_{v+w}^\infty \mathbf{P}_N^0(T_1 > u, Z_0 = 1)du.
\end{aligned}
$$

The same computation applies for the ongoing idle period.

Using (9), it is also easy to establish that

$$
\mathbf{P}(Z_0 = 0) = \frac{\mathbf{E}_N^0[T_1|Z_0 = 0]}{\mathbf{E}_N^0[T_1|Z_0 = 0] + \mathbf{E}_N^0[T_1|Z_0 = 1]},
$$

whereas (8) yields that

$$
\mathbf{P}_N^0(Z_0 = 0) = \frac{1}{2}.
$$

At the starting times of busy periods of S, both storages are empty (see Figure 1). It follows that the busy cycles of S, consisting of a busy period followed by an idle period, are independent and identically distributed. This holds not only for their lengths but for the whole paths. Moreover each idle period is independent from the following busy period. The remaining pair of interest, a busy period and the following idle period, are probably dependent, but at present we don't have an exact argument showing this.

Note that each busy period ends with an interval where the queue length decreases linearly. This shows that the process S is not reversible in time (unlike X).

5 Stationary distribution

The next proposition is fundamental in showing that the storage occupancy process S as defined above is a meaningful and interesting object of study. In fact, the case with general time homogeneous diffusions has already been treated in [20], from which the result below follows as a special case. We repeat the argument here because of the key importance of the result for this paper, and also to make the presentation more self-contained.

Proposition 5.1 *The process S is a stationary process in stationary state if and only if $0 < \mu < 1$. The stationary distribution is given (for all $t \in \mathbb{R}$) by*

$$\mathbf{P}(S_t > a) = \mu\, \mathrm{e}^{-2(1-\mu)a}, \quad a \geq 0,$$

and, consequently, $\mathbf{P}(S_t = 0) = 1 - \mu$.

Proof From the description (3) it is seen that

$$S_0 := \sup_{s \leq 0}\{-L_s + s\} \overset{(\mathrm{d})}{=} \sup_{s \geq 0}\{L_s - s\}.$$

We compute the distribution of $\sup_{s\geq 0}\{L_s - s\}$ given that $X_0 = 0$. Let $\mathbf{P}_0$ denote the probability measure associated to $\{X_t : t \geq 0\}$ when started from 0. Introduce the right continuous inverse A of L by setting for $s \geq 0$

$$A_s := \inf\{t : L_t > s\}.$$

Then, as is well known (see [12] or [7]), A is a subordinator and its Lévy–Khintchine representation is

$$\begin{aligned} \mathbf{E}_0(\mathrm{e}^{-\alpha A_t}) &= \exp\big(-t\int_0^\infty (1-\mathrm{e}^{-\alpha u})\frac{1}{\sqrt{2\pi u^3}}\,\mathrm{e}^{-\frac{\mu^2}{2}u}du\big) &(10)\\ &= \exp\big(-t(\sqrt{2\alpha+\mu^2}-\mu)\big). &(11)\end{aligned}$$

Because A is the right continuous inverse of L we have

$$\{L_t - t < a \quad \forall\, t \geq 0\} = \{A_t - t > -a \quad \forall\, t \geq 0\}.$$

Hence, we study the process $\{A_t - t : t \geq 0\}$ which is a spectrally positive Lévy process with (cf. (7))

$$\mathbf{E}_0(\mathrm{e}^{-\alpha(A_t - t)}) = \exp\big(t\,(\alpha - \sqrt{2\alpha+\mu^2}+\mu)\big). \tag{12}$$

Introduce for $\alpha \geq 0$

$$\psi(\alpha) := \alpha - \sqrt{2\alpha+\mu^2} + \mu. \tag{13}$$

Then $\psi(0)=0$, and it is easily seen that if $\mu\geq 1$ then ψ is increasing. Further, if $0<\mu<1$ there exists a unique $\alpha^*=2(1-\mu)>0$ such that $\psi(\alpha^*)=0$ and ψ is increasing for $\alpha>\alpha^*$. Let for $a>0$

$$T_a:=\inf\{t:A_t-t=-a\}.$$

Then (see [5] or [6])

$$\mathbf{E}_0\big(\mathrm{e}^{-\beta T_a};T_a<\infty\big)=\mathrm{e}^{-a\,\eta(\beta)},$$

where η is the inverse of $\alpha\mapsto\psi(\alpha),\ \alpha\geq\alpha^*$. Consequently,

$$\begin{aligned}\mathbf{P}_0(\sup_{t\geq 0}\{L_t-t\}<a) &= \mathbf{P}_0(L_t-t<a\quad\forall\, t\geq 0)\\ &= \mathbf{P}_0(A_t-t>-a\quad\forall\, t\geq 0)\\ &= \mathbf{P}_0\big(T_a=\infty\big)=1-\mathrm{e}^{-a\,\eta(0)}.\end{aligned}$$

In the case ψ is increasing, i.e., $\mu\geq 1$, we have $\eta(0)=0$ which means that $\sup_{t\geq 0}\{L_t-t\}=\infty$ with probability 1. On the other hand, when $0<\mu<1$ we have $\eta(0)=2(1-\mu)$ and, hence,

$$\mathbf{P}_0(\sup_{t\geq 0}\{L_t-t\}>a)=\mathrm{e}^{-2(1-\mu)a}. \tag{14}$$

The distribution of $\sup_{t\geq 0}\{L_t-t\}$ given that $X_0=y>0$ is obtained by the strong Markov property. Indeed, denoting $H_0:=\inf\{t\geq 0:X_t=0\}$, we have for $a>0$

$$\begin{aligned}\mathbf{P}_y(\sup_{t\geq 0}\{L_t-t\}<a) &= \mathbf{P}_y(L_{H_0+t}-(H_0+t)<a\quad\forall\, t\geq 0)\\ &= \int_0^\infty \mathbf{P}_y(H_0\in du)\mathbf{P}_0(L_t-t<a+u\quad\forall\, t\geq 0)\\ &= \int_0^\infty \mathbf{P}_y(H_0\in du)\big(1-\mathrm{e}^{-2(1-\mu)(a+u)}\big)\\ &= 1-\mathrm{e}^{-2(1-\mu)a}\mathbf{E}_y\big(\mathrm{e}^{-2(1-\mu)H_0}\big)\\ &= 1-\mathrm{e}^{-2(1-\mu)(a+y)},\end{aligned}$$

where the formula (see [7])

$$\mathbf{E}_y\big(\mathrm{e}^{-\alpha H_0}\big)=\mathrm{e}^{-(\sqrt{2\alpha+\mu^2}-\mu)y} \tag{15}$$

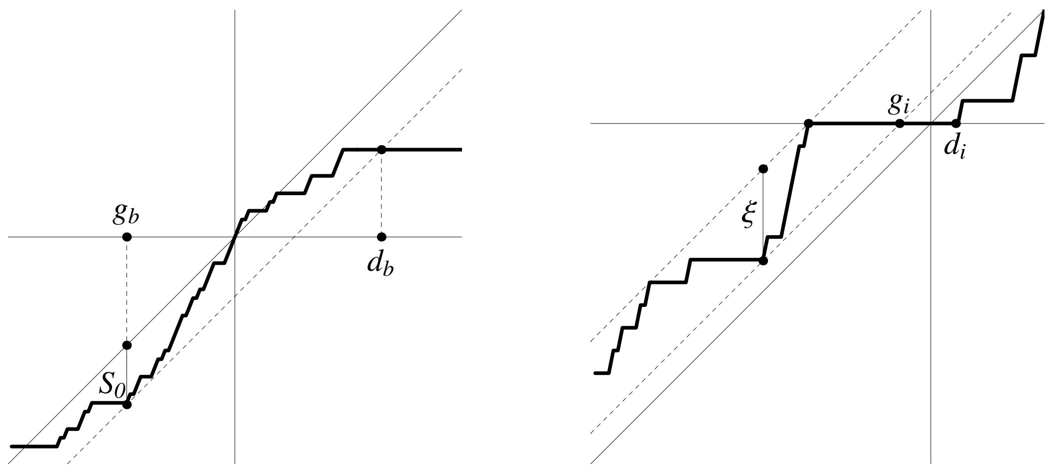

Figure 2: Illustration for some notation in the proofs. The thick line denotes a realization of the process L_t. Left: busy period at time 0. Right: idle period at time 0.

is used. Integrating with respect to the stationary distribution m gives, in the case $0 < \mu < 1$,

$$\begin{aligned}
\mathbf{P}(S_0 > a) &= \mathbf{P}(\sup_{t\geq 0}\{L_t - t\} > a) \\
&= \int_0^\infty m(dy)\mathrm{e}^{-2(1-\mu)(a+y)} \\
&= \mu\,\mathrm{e}^{-2(1-\mu)a}
\end{aligned}$$

as claimed. From Proposition 2.2 it follows that $S_t \sim S_0$ for all t and that $S_t - S_s \sim S_{t-s} - S_0$ for all $t > s$. This completes the proof. □

6 Idle periods

In this section we compute the joint distribution of the starting time and the ending time of the on-going idle period. From this the distribution of the length of the on-going idle period is easily obtained.

Definition 6.1 *The random variables*

$$g_i := \sup\{t < 0 : S_t > 0\} \quad \text{and} \quad d_i := \inf\{t > 0 : S_t > 0\} \tag{16}$$

are called, in case $S_0 = 0$, *the starting time and the ending time, respectively, of the on-going idle period (observed at time 0).*

We know from Section 4 (although S is not reversible) that

$$|g_i| \sim d_i.$$

Before stating and proving the main result, Proposition 6.4, some preliminary observations are made. Firstly, from Proposition 5.1 or from Little's law, the probability for an idle period at time 0 is

$$\mathbf{P}(S_0 = 0) = \mathbf{P}(\sup_{u\geq 0}\{L_u - u\} = 0) = 1 - \mu.$$

Let

$$H_0^{(1)} := -\sup\{t < 0 \,:\, X_t = 0\} = \inf\{t > 0 \,:\, X_t^{(1)} = 0\}$$

and

$$H_0^{(2)} := \inf\{t > 0 \,:\, X_t = 0\} = \inf\{t > 0 \,:\, X_t^{(2)} = 0\}.$$

As computed in the proof of Proposition 5.1, see (14) with the opposite direction of time,

$$\xi := \sup_{t\leq 0}\{-L_{-H_0^{(1)}+t} + t\} \tag{17}$$

is exponentially distributed with parameter $\alpha^* := 2(1-\mu)$. From Figure 2 it is seen that $\{\xi < H_0^{(1)}\} = \{S_0 = 0\}$. By the strong Markov property of $X^{(1)}$, the variables ξ and $H_0^{(1)}$ are independent giving

$$\mu = \mathbf{P}(S_0 > 0) = \mathbf{P}(\xi > H_0^{(1)}) = \mathbf{E}(\mathrm{e}^{-2(1-\mu)H_0^{(1)}}).$$

We remark that the formula

$$\mathbf{E}(\mathrm{e}^{-2(1-\mu)H_0^{(1)}}) = \mu$$

is obtained also from (15) by integrating with respect to m. Further, from Figure 2,

$$g_i = \sup\{s < 0 : \sup_{t\leq 0}\{-L_{s+t} + t\} > 0\},$$

and, consequently, we have

Proposition 6.2 *The conditional law of $|g_i|$ given that $S_0 = 0$ is the same as the conditional law of $H_0^{(1)} - \xi$ given that $\xi < H_0^{(1)}$.*

Consider next the variable d_i, the ending time of the on-going idle period.

Proposition 6.3 *Given that $S_0 = 0$ the variables d_i and $H_0^{(2)}$ are a.s. equal.*

Proof Clearly, see Figure 2, $H_0^{(2)} \le d_i$ and the claim follows if $\{L_t - t : t \ge 0\}$ is a.s. initially increasing under $\mathbf{P}_0$, i.e.,

$$\mathbf{P}_0(\exists\, \varepsilon > 0 \text{ such that } L_t - t > 0 \quad \forall\, t \in (0,\varepsilon)\} = 1. \tag{18}$$

To verify this, notice that $\sup_{t\ge 0}\{L_t - t\} \ge 0$ and that for all $x > 0$

$$\mathbf{P}_x(\sup_{t\ge 0}\{L_t - t\} = 0) > 0. \tag{19}$$

If (18) does not hold then there exist $\varepsilon > 0$ such that

$$\mathbf{P}_0(L_t - t < 0 \quad \forall\, t \in (0,\varepsilon)) > 0. \tag{20}$$

Combining (20) with (19) gives

$$\begin{aligned}
&\mathbf{P}_0(\sup_{t\ge 0}\{L_t - t\} = 0) \\
&\qquad \ge \mathbf{E}_0\big(\mathbf{P}_{X_\varepsilon}(\sup_{t\ge 0}\{L_t - t\} = 0)\,;\, L_t - t < 0 \quad \forall\, t \in (0,\varepsilon)\big) > 0,
\end{aligned}$$

which contradicts (14). □

We are now ready to prove the main result of this section.

Proposition 6.4 *The Laplace transform of the joint distribution of g_i and d_i given that $S_0 = 0$ is*

$$\begin{aligned}
&\mathbf{E}\big(\mathrm{e}^{\alpha g_i - \beta d_i}\,|\,S_0 = 0\big) \\
&\quad = \mathbf{E}\big(\mathrm{e}^{-\alpha(H_0^{(1)} - \xi) - \beta H_0^{(2)}}\,|\,H_0^{(1)} > \xi\big) \\
&\quad = \frac{8\mu}{\big(\sqrt{2\alpha+\mu^2}+2-\mu\big)\big(\sqrt{2\beta+\mu^2}+2-\mu\big)\big(\sqrt{2\alpha+\mu^2}+\sqrt{2\beta+\mu^2}\big)}.
\end{aligned}$$

Hence,

$$\mathbf{E}\big(\mathrm{e}^{\alpha g_i}\,|\,S_0=0\big) \;=\; \mathbf{E}\big(\mathrm{e}^{-\alpha d_i}\,|\,S_0=0\big)=\frac{2\mu}{\sqrt{2\alpha+\mu^2}+\alpha+\mu},$$

and

$$\mathbf{E}\big(-g_i\,|\,S_0=0\big)=\mathbf{E}\big(d_i\,|\,S_0=0\big)=\frac{1+\mu}{2\mu^2}.$$

Proof Using the conditional independence of $X^{(1)}$ and $X^{(2)}$ we obtain

$$\begin{aligned}&\mathbf{E}\big(\mathrm{e}^{-\alpha(H_0^{(1)}-\xi)-\beta H_0^{(2)}}\,;H_0^{(1)}>\xi\big)\\ &\qquad=\int_0^\infty 2\mu\mathrm{e}^{-2\mu x}\mathbf{E}_x\big(\mathrm{e}^{-\beta H_0^{(2)}}\big)\mathbf{E}\big(\mathrm{e}^{-\alpha(H_0^{(1)}-\xi)}\,;H_0^{(1)}>\xi\big).\end{aligned}$$

We have

$$\mathbf{E}_x\big(\mathrm{e}^{-\beta H_0^{(2)}}\big)=\mathrm{e}^{-(\sqrt{2\beta+\mu^2}-\mu)x}$$

and

$$\mathbf{E}_x\big(\mathrm{e}^{-\alpha(H_0^{(1)}-\xi)}\,;H_0^{(1)}>\xi\big)=\int_0^\infty \mathbf{P}_x(H_0^{(1)}\in dt)\,\mathrm{e}^{-\alpha t}\mathbf{E}_0\big(\mathrm{e}^{\alpha\xi}\,;\xi<t\big).$$

Further, because $\xi\sim\mathrm{Exp}(\alpha^*),\ \alpha^*:=2(1-\mu)$,

$$\mathbf{E}_0\big(\mathrm{e}^{\alpha\xi}\,;\xi<t\big)=\int_0^t \mathrm{e}^{\alpha u}\alpha^*\mathrm{e}^{-\alpha^* u}du \;=\; \frac{\alpha^*}{\alpha^*-\alpha}\big(1-\mathrm{e}^{-(\alpha^*-\alpha)t}\big).$$

Putting the pieces together and integrating give

$$\begin{aligned}&\mathbf{E}\big(\mathrm{e}^{-\alpha(H_0^{(1)}-\xi)-\beta H_0^{(2)}}\,;H_0^{(1)}>\xi\big)\\ &\quad=\frac{\alpha^*}{\alpha^*-\alpha}\int_0^\infty 2\mu\Big(\mathrm{e}^{-(\sqrt{2\alpha+\mu^2}-\sqrt{2\beta+\mu^2})x}-\mathrm{e}^{-(\sqrt{2\beta+\mu^2}+2-\mu)x}\Big)\,dx\\ &\quad=\frac{2(1-\mu)}{2(1-\mu)-\alpha}\Big(\frac{2\mu}{\sqrt{2\alpha+\mu^2}+\sqrt{2\beta+\mu^2}}-\frac{2\mu}{\sqrt{2\beta+\mu^2}+2-\mu}\Big).\end{aligned}$$

Dividing this expression with

$$1-\mu=\mathbf{P}(S_0=0)=\mathbf{P}(H_0^{(1)}>\xi)$$

leads after some manipulations to the desired Laplace transform. The proofs of the remaining claims being straightforward are left to the reader. □

Remark 6.5 Notice that

$$\lim_{\mu\to 1}\mathbf{E}\big(H_0^{(1)}-\xi\,|\,H_0^{(1)}>\xi\big) \;=\; \lim_{\mu\to 1}\mathbf{E}\big(H_0^{(2)}\,|\,H_0^{(1)}>\xi\big)=1.$$

7 Busy periods

In this section we focus on busy periods of the storage process S and find the Laplace transform of the distribution of the starting time of the on-going busy period. Recall from Section 4 that the starting time and the ending time are identical in law. Recall from Proposition 5.1 that the probability that there is an busy period at time 0 is μ, i.e.,

$$\mathbf{P}(S_0 > 0) = \mu.$$

Definition 7.1 *The random variables*

$$g_b := \sup\{t < 0 : S_t = 0\} \quad \text{and} \quad d_b := \inf\{t > 0 : S_t = 0\} \tag{21}$$

are called, in case $S_0 > 0$, the starting time and the ending time, respectively, of the on-going busy period (observed at time 0). (See Figure 2.)

For the starting time of the busy period we have

$$\begin{aligned} g_b &= \sup\{t < 0 : S_t = 0\} \\ &= \sup\{t < 0 : \sup_{s \le t}\{L_t - L_s - (t-s)\} = 0\} \\ &= \sup\{t < 0 : L_t - t = -\sup_{s \le t}\{-L_s + s\}\} \\ &= \sup\{t < 0 : L_t - t = -\sup_{s \le 0}\{-L_s + s\}\} \\ &= \sup\{t < 0 : L_t - t = S_0\}. \end{aligned}$$

When computing the distribution of g_b it is convenient to reverse time. In other words, let

$$M := \sup_{t \ge 0}\{L_t - t\}$$

then

$$|g_b| \overset{(d)}{=} T_M := \inf\{t \ge 0 : L_t - t = M\},$$

and, therefore, we find the distribution of T_M.

Proposition 7.2 *The $\mathbf{P}_0$-distribution of T_M (that is, given that $X_0 = 0$) has the Laplace transform*

$$\mathbf{E}_0(\mathrm{e}^{-\alpha T_M}) = \frac{2(1-\mu)}{\sqrt{2\alpha + (1-\mu)^2} + (1-\mu)}$$

Remark 7.3 *From (15) when integrating with respect to m it is seen that*

$$\mathbf{E}(\mathrm{e}^{-\alpha H_0}) = \frac{2\mu}{\sqrt{2\alpha+\mu^2}+\mu}.$$

Therefore, the $\mathbf{P}_0$*–distribution of* T_M *is the same as the distribution of* H_0 *for RBM*$(-(1-\mu))$ *in stationary state.*

Proof As in the proof of Proposition 5.1, let A be the right continuous inverse of L and consider the spectrally positive Lévy process $\{A_t - t\, :\, t\geq 0\}$. We have $A_t - t \to +\infty$ as $t\to+\infty$ and, hence,

$$N := \inf\{A_t - t\, :\, t\geq 0\} > -\infty.$$

Define

$$T_N := \inf\{t>0\, :\, A_{t-} - t = N\}.$$

Because A is the inverse of L, we have

$$M = -N \quad \text{and} \quad T_M = N + T_N.$$

The next step is to find the law of (N, T_N). For this we use the result in Bertoin [4]. Recall from (12) that

$$\mathbf{E}(\mathrm{e}^{-\alpha(A_t - t)}) = \exp\bigl(t\psi(\alpha)\bigr),$$

where

$$\psi(\alpha) := \alpha - \sqrt{2\alpha+\mu^2}+\mu.$$

Let for $\alpha\geq 0$

$$\psi^{\downarrow}(\alpha) := \psi(\alpha+\alpha^*) = \alpha - \sqrt{2\alpha+(2-\mu)^2}+2-\mu. \tag{22}$$

Define further $B^{\downarrow}$ to be the spectrally positive Lévy process such that

$$\mathbf{E}(\mathrm{e}^{-\alpha B^{\downarrow}_t}) = \exp\bigl(t\psi^{\downarrow}(\alpha)\bigr).$$

The basic fact about $B^{\downarrow}$ is that $B^{\downarrow}_t \to -\infty$ a.s. as $t\to\infty$. Let $\mathbf{P}^{\times}$ denote the product measure which governs the process $B^{\downarrow}$, $B^{\downarrow}_0 = 0$, and an (independent) exponentially α^*–distributed random variable E. Then (see Bertoin [4]) the pre–T_N–process $\{A_t - t\ :\ 0\leq t < T_N\}$ is identical in law to $\{B^{\downarrow}_t\ :\ 0\leq t< H_{-E}\}$ where

$$H_{-E} := \inf\{t\ :\ B^{\downarrow}_t = -E\}.$$

The Laplace–transform of H_a, $a<0$, is given in terms of the inverse $\eta^{\downarrow}$ of $\psi^{\downarrow}$ (cf. the proof of Proposition 5.1):

$$\mathbf{E}^{\downarrow}(\mathrm{e}^{-\alpha H_a}) = \exp\big(-a\eta^{\downarrow}(\alpha)\big),$$

where

$$\eta^{\downarrow}(\alpha) := \alpha + \sqrt{2\alpha+(1-\mu)^2} - (1-\mu). \tag{23}$$

Now we are ready to compute

$$\begin{aligned}
\mathbf{E}_0(\mathrm{e}^{-\alpha T_M}) &= \mathbf{E}(\mathrm{e}^{-\alpha(N+T_N)}) \\
&= \mathbf{E}^{\times}(\mathrm{e}^{-\alpha(-E+H_{-E})}) \\
&= \int_0^\infty \mathrm{e}^{\alpha u}\, \mathbf{E}^{\downarrow}(\mathrm{e}^{-\alpha H_{-E}}\,|\,E=u)\,\mathbf{P}(E\in du) \\
&= \int_0^\infty \mathrm{e}^{\alpha u}\mathrm{e}^{-u\eta^{\downarrow}(\alpha)}\,\alpha^*\mathrm{e}^{-\alpha^* u}du \\
&= \frac{\alpha^*}{\eta^{\downarrow}(\alpha)+\alpha^*-\alpha} \\
&= \frac{2(1-\mu)}{\sqrt{2\alpha+(1-\mu)^2}+(1-\mu)},
\end{aligned}$$

as claimed. □

In Proposition 7.2 it is assumed that the underlying RBM($-\mu$) starts at 0 and, hence, the local time increases initially. However, we are, in fact, interested in the case when the RBM($-\mu$) is in stationary state and the local time starts to increase at H_0. This means that the starting value A_0 of the process A is given by

$$A_0 = \inf\{t : L_t > 0\} = H_0.$$

Let (cf. (17))

$$\hat{\xi} := \sup_{s\geq 0}\{L_{H_0+s} - s\}$$

and define

$$T_{\hat{\xi}} := \inf\{t\geq 0 : L_{H_0+t} - t = \hat{\xi}\}$$

Then $T_{\hat{\xi}}$ and H_0 are independent and, in the case $M>0$,

$$T_M = T_{\hat{\xi}} + H_0.$$

It is clear that the $\mathbf{P}$-distribution of $T_{\hat{\xi}}$ is the same as the $\mathbf{P}_0$-distribution of T_M. Consequently, we obtain

$$\begin{aligned}
\mathbf{E}(\mathrm{e}^{-\alpha T_M}; M>0) &= \mathbf{E}\big(\mathrm{e}^{-\alpha(T_{\hat{\xi}}+H_0)}; \hat{\xi} > H_0\big) \\
&= \int_0^\infty \mathbf{E}\big(\mathrm{e}^{-\alpha(T_{\hat{\xi}}+H_0)}; \hat{\xi} > H_0 \,|\, H_0 = t\big)\, \mathbf{P}(H_0 \in dt) \\
&= \int_0^\infty \mathrm{e}^{-\alpha t} \mathbf{E}_0\big(\mathrm{e}^{-\alpha T_M}; M > t\big)\, \mathbf{P}(H_0 \in dt) \\
&= \int_0^\infty \mathrm{e}^{-\alpha t} \Big(\int_t^\infty \mathrm{e}^{\alpha u}\, \mathrm{e}^{-u\,\eta^{\downarrow}(\alpha)}\, \alpha^* \mathrm{e}^{-\alpha^* u} du\Big) \mathbf{P}(H_0 \in dt) \\
&= \frac{\alpha^*}{\eta^{\downarrow}(\alpha)+\alpha^*-\alpha} \int_0^\infty \mathrm{e}^{-(\eta^{\downarrow}(\alpha)+\alpha^*)t}\ \mathbf{P}(H_0 \in dt) \\
&= \frac{\alpha^*}{\eta^{\downarrow}(\alpha)+\alpha^*-\alpha}\ \frac{2\mu}{\sqrt{2(\eta^{\downarrow}(\alpha)+\alpha^*)+\mu^2}+\mu} \\
&= \frac{\alpha^*}{\eta^{\downarrow}(\alpha)+\alpha^*-\alpha}\ \frac{2\mu}{\sqrt{2\eta^{\downarrow}(\alpha)+(2-\mu)^2}+\mu}.
\end{aligned}$$

Using

$$\alpha = \psi^{\downarrow}(\eta^{\downarrow}(\alpha))$$

gives (cf. (22))

$$\sqrt{2\eta^{\downarrow}(\alpha)+(2-\mu)^2} = \eta^{\downarrow}(\alpha) - \alpha + 2 - \mu.$$

But (cf. (23))

$$\eta^{\downarrow}(\alpha) := \alpha + \sqrt{2\alpha+(1-\mu)^2} - (1-\mu),$$

and so

$$\begin{aligned}
\mathbf{E}(\mathrm{e}^{-\alpha T_M}; M>0) &= \frac{2(1-\mu)}{\sqrt{2\alpha+(1-\mu)^2}+1-\mu}\ \frac{2\mu}{\sqrt{2\alpha+(1-\mu)^2}+1+\mu} \\
&= \mu\ \frac{2(1-\mu)}{\sqrt{2\alpha+(1-\mu)^2}+\alpha+1-\mu}.
\end{aligned}$$

Hence remarking that

$$\mathbf{P}(g_b < 0) = \mathbf{P}(M > 0) = \mathbf{P}(S_0 > 0) = \mathbf{P}(\hat{\xi} > H_0) = \mu$$

we have the following

Proposition 7.4 *The common Laplace transform of the distributions of the starting time and the ending time of the on-going busy period is*

$$\begin{aligned}\mathbf{E}(\mathrm{e}^{\alpha g_b}\,|\,S_0>0) &= \mathbf{E}(\mathrm{e}^{-\alpha d_b}\,|\,S_0>0)\\ &= \frac{2(1-\mu)}{\sqrt{2\alpha+(1-\mu)^2}+\alpha+1-\mu}\end{aligned}$$

Remark 7.5 Notice from Proposition 6.4 the nice feature that the distributions of g_i and d_i are of the same form as the distributions of g_b and d_b. The parameter μ in the first case corresponds to $1-\mu$ in the second case. In fact, this similarity extends also to their joint distributions, as will be proved in a forthcoming paper.

8 Multifractal analysis of the input process

The set of growth points of the continuous process L has almost surely Lebesgue measure zero. Multifractal analysis is a technique for studying the fine structure of the corresponding singular measure $\nu_L((s,t]) = L_t - L_s$. It turns out that this fine structure is independent of the system parameter μ.

Let ν be a measure on $\mathbb{R}$ and consider the lower and upper pointwise dimensions of ν at x:

$$\begin{aligned}\underline{h}(\nu,x) &:= \liminf_{r\downarrow 0}\frac{\log\nu(x-r,x+r)}{\log r},\\ \overline{h}(\nu,x) &:= \limsup_{r\downarrow 0}\frac{\log\nu(x-r,x+r)}{\log r},\end{aligned}$$

correspondingly. The distribution of the local scaling laws can be described by the multifractal spectra

$$\begin{aligned}\underline{f}(\nu,h) &:= \dim\{x\in\operatorname{supp}\nu : \underline{h}(\nu,x)=h\},\\ \overline{f}(\nu,h) &:= \dim\{x\in\operatorname{supp}\nu : \overline{h}(\nu,x)=h\},\end{aligned}$$

where dim denotes the Hausdorff dimension (with $\dim\emptyset := -\infty$).

A wide class of local time processes can be represented as the occupation measure of subordinators. Let us consider the measure ν determined by a subordinator A on $[0,1]$:

$$\nu(B) = |\{t\in[0,1] : A_t\in B\}|. \tag{24}$$

Denote the random set of values attained by the process A on $[0,1]$ by $S = \{A_t : t \in [0,1]\}$. If ν corresponds to the local time at 0 of the process X, then $S = \{\tau \in [0,A_1] : X_\tau = 0\}$. Since A is a subordinator, there exists a function g (so-called Laplace exponent) such that

$$\mathbf{E}\mathrm{e}^{-rA_t} = \mathrm{e}^{-tg(r)}.$$

Define

$$\begin{aligned} \sigma &= \liminf_{r\to\infty} \frac{\log g(r)}{\log r}, \\ \beta &= \limsup_{r\to\infty} \frac{\log g(r)}{\log r}. \end{aligned}$$

The multifractal structure of the measures having the form (24) has been determined by Hu and Taylor [10, 11]. Assuming that there exists a number $\lambda > 1$ such that

$$1 < \liminf_{x\to\infty} \frac{g(\lambda x)}{g(x)} \le \limsup_{x\to\infty} \frac{g(\lambda x)}{g(x)} < \lambda,$$

then, for almost every sample path,

$$\underline{h}(\nu,x) = \sigma \qquad \forall x \in S, \tag{25}$$

$$\beta \le \overline{h}(\nu,x) \le 2\beta \qquad \forall x \in S. \tag{26}$$

The first equality implies that $\underline{f}(\nu,\sigma) = \dim S = \sigma$. Moreover, Hu and Taylor showed that, for almost every sample path,

$$\dim\left\{ t \in [0,1] : \limsup_{r\downarrow 0} \frac{\log \nu(A_t - r, A_t + r)}{-\log g(1/r)} = h \right\} = \begin{cases} \dfrac{2}{h} - 1, & \text{if } h \in [1,2] \\ -\infty, & \text{otherwise.} \end{cases}$$

Proposition 8.1 *Consider the system conditioned on $X_0 = 0$. For almost every sample path of L,*

$$\underline{h}(\nu_L,x) = \frac{1}{2} \quad \textit{for all } x \in S,$$

$$\frac{1}{2} \le \overline{h}(\nu_L,x) \le 1 \quad \textit{for all } x \in S.$$

Moreover, almost surely,

$$\begin{aligned} \underline{f}(\nu_L,\tfrac{1}{2}) &= \frac{1}{2}, \\ \overline{f}(\nu_L,h) &= \frac{1}{2h} - \frac{1}{2}, \quad \textit{if } \frac{1}{2} \le h \le 1. \end{aligned}$$

Proof The measure ν_L defined by the local time process L can be represented as the occupation measure of a subordinator A with the Laplace exponent $g(r) = \sqrt{2r+\mu^2} - \mu$ (see (11) in section 5). Thus, $\sigma = \beta = \frac{1}{2}$, and (25) and (26) give the ranges of the scaling laws.

Since

$$\limsup_{r\downarrow 0} \frac{\log \nu(A_t - r, A_t + r)}{-\log g(1/r)} = \limsup_{r\downarrow 0} \frac{\log \nu(A_t - r, A_t + r)}{\frac{1}{2}\log r},$$

we have

$$\dim\left\{t \in [0,1] : \limsup_{r\downarrow 0} \frac{\log \nu(A_t - r, A_t + r)}{\log r} = h\right\} = \frac{1}{h} - 1$$

for $h \in [\frac{1}{2}, 1]$. To complete the proof, recall that for any subordinator A the inequalities

$$\sigma \dim E \leq \dim\{A_t : t \in E\} \leq \beta \dim E$$

hold for all subsets $E \subset [0,\infty)$, a.s. (see [9]). □

Using the results by Jaffard [13, 14], we can also calculate the spectrum of the subordinator related to the local time process L.

Theorem 8.2 *[13, 14] Let Y be an increasing Lévy process with Lévy measure Π. Denote*

$$C_j = \Pi([2^{-j-1}, 2^{-j}]), \quad j = 1, 2, \ldots$$

and

$$\gamma = \sup\left(0, \limsup_{j\to\infty} \frac{\log C_j}{j \log 2}\right) = \inf\left\{\alpha \geq 0 : \int_0^1 x^\alpha \Pi(dx) < \infty\right\}.$$

Assume further that $\gamma > 0$ and $\sum_{j=1}^{\infty} 2^{-j}\sqrt{C_j \log(1+C_j)} < \infty$. Then, for almost every sample path of Y, the multifractal spectrum of the process Y is given by

$$\dim\{t : H_Y(t) = h) = \begin{cases} \gamma h, & \textit{if } 0 \leq h \leq \frac{1}{\gamma}, \\ -\infty, & \textit{otherwise}, \end{cases}$$

where $H_Y(t)$ denotes the pointwise Hölder exponent[1] *at t.*

[1] A function f is in C_t^h if there is a polynomial $u \mapsto P_t(u)$ such that $|f(u) - P_t(u)| \leq C|u-t|^h$ for u sufficiently close to t. The pointwise Hölder exponent of f at t is $H_f(t) = \sup\{h : f \in C_t^h\}$.

Corollary 8.3 *For almost every path, the lower multifractal spectrum of the subordinator A with the Laplace exponent* $g(r) = \sqrt{2r+\mu^2} - \mu$ *is*

$$\underline{f}(\nu_A, h) = \begin{cases} \frac{1}{2}h, & \textit{if } 0 \le h \le 2, \\ -\infty, & \textit{otherwise}. \end{cases}$$

Proof By (10) in Section 5

$$\Pi(dx) = \frac{\mathrm{e}^{-\frac{\mu^2}{2}x}}{\sqrt{2\pi x^3}}\mathrm{d}x.$$

It is straightforward to see that the numbers C_j of Theorem 8.2 satisfy

$$C_j \in \left[\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\mu^2 2^{-j-2}} 2^{j/2}, \sqrt{\frac{2}{\pi}}\mathrm{e}^{-\mu^2 2^{-j-1}} 2^{j/2}\right],$$

which entails that $\gamma = \frac{1}{2} > 0$ and

$$\sum_{j=0}^{\infty} 2^{-j}\sqrt{C_j \log(1+C_j)} \le \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\infty} 2^{-j/2} < \infty.$$

Moreover, the pointwise Hölder exponent $H_A(t)$ and the lower local dimension $\underline{h}(\nu_A, t)$ are equal since the Lévy process A is increasing and has no drift.

□

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