Encyclopaedia of Mathematics, Supplement II, p. 94-95,
ed. M. Hazewinkel, Kluwer Academic Publishers, Dordrecht (2000).

Brownian local time

by

Paavo Salminen

Åbo Akademi University, Mathematical Department,

FIN-20500 Åbo, Finland

Let $W = \{W_t : t \geq 0\}$ be a standard Wiener process (or, in other words, **Brownian motion**) living on $\mathbf{R}$ and started at 0. The random set $\mathcal{Z}_0 := \{t : W_t = 0\}$, the so called *zero set of the Brownian path*, is a.s. (=almost surely) **perfect** (i.e. closed and dense in itself), unbounded and of **Lebesgue measure** 0. The complement of $\mathcal{Z}_0$ is a countable union of open intervals.

The remarkable result of Paul Lévy ([6], [7]) is that there exists a non-decreasing (random) function determined by $\mathcal{Z}_0$ which is constant on the open intervals in the complement of $\mathcal{Z}_0$ and which has every point in $\mathcal{Z}_0$ as a (left and/or right) strict increase point. This function is called the *Brownian local time* (at 0). It is clear that a similar construction can be made at any point x.

The existence of the local time can be deduced from the fact (due to Lévy) that the processes $W^+ := \{|W_t| : t \geq 0\}$ and $W^o := \{M_t - W_t : t \geq 0\}$, where $M_t := \sup_{s \leq t} W_s$, are identical in law. Indeed, for W^o the function $t \mapsto M_t$ has the desired properties of local time; for the proof that M_t for a given t is determined by $\mathcal{Z}_0^o(t) := \{s : M_s - W_s = 0,\ s \leq t\}$ see Itô and McKean [4]. Because W^+ and W^o are identical in law there exists a function with corresponding properties connected to W^+.

Let $\ell(t,x)$ be the Brownian local time at x at time t. Then we have a.s.

$$\ell(t,x) = \lim_{\epsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{(x-\varepsilon, x+\varepsilon)}(W_s)\, ds,$$

and this leads to the occupation time formula

$$\int_0^t f(W_s)\, ds = \int \ell(t,x) f(x)\, dx,$$

where f is a Borel-measurable function.

As seen above $\ell(t,0)$ can be viewed as the measure of the zero set $\mathcal{Z}_0 \cap [0,t]$. In fact, it has been proved by Taylor and Wendel [11] and Perkins [9] that ℓ is the random **Hausdorff l-measure** of $\mathcal{Z}_0 \cap [0,t]$ with $l(u) = (2u|\ln|\ln u||)^{1/2}$.

Introduce for $x > 0$ the right-continuous inverse of M :

$$\tau_x := \inf\{s : M_s > x\}.$$

By the **strong Markov property** and spatial homogeneity of Brownian motion the process $\tau := \{\tau_x : x \geq 0\}$ is increasing and has independent and identically distributed increments, in other words, τ is a subordinator. Because $\ell(t,0)$ and M_t are for every $t \geq 0$ identical in law also the so called inverse local time

$$\alpha_x := \inf\{s : \ell(s,0) > x\}.$$

and τ_x are identical in law. Hence, the finite dimensional distributions of α are determined by the Laplace transform

$$\begin{aligned}\mathbf{E}\big(\exp(-u\,\alpha_x)\big) &= \exp\big(-x\int_0^\infty (1-e^{-uv})\frac{1}{\sqrt{2\pi v^3}}dv\big)\\ &= \exp(-x\sqrt{2u}).\end{aligned}$$

The mapping $(t,x) \mapsto \ell(t,x),\ t \geq 0, x \in \mathbf{R}$, is continuous. This is due to Trotter [12]; for a proof based on Itô's formula see, e.g., Ikeda and Watanabe [3].

The behaviour of the process $\{\ell(T,x) : x \in \mathbf{R}\}$ can be characterized for some stopping times T (for first hitting times, for instance). Results in this direction are called **Ray-Knight theorem**s [10],[5], see also Borodin and Salminen [2].

The process $\{\ell(t,0) : t \geq 0\}$ is an example of an additive functional of Brownian motion having support at one point (i.e. at 0). As such it is unique up to a multiplicative constant. See Blumenthal and Getoor [1].

Brownian local time is an important concept both in theory and in applications of stochastic processes. It can be used, e.g., to construct diffusions from Brownian motion via random time change and to analyze stochastic differential equations. There are some natural problems in stochastic optimal control (finite fuel problem) and in financial mathematics (barrier options), for instance, where (Brownian) local time plays a crucial role.

For a survey article on Brownian local time see McKean [8].

References

[1] Blumenthal, R.M., Getoor, R.K.: Markov Processes and Potential Theory. Academic Press, New York, London 1968.

[2] Borodin, A.N., Salminen P.: Handbook of Brownian Motion – Facts and Formulae. Birkhauser Verlag, Basel, Boston, Berlin, 1996.

[3] Ikeda, N., Watanabe, S.: Stochastic Differential Equations and Diffusion Processes. North-Holland Publ. Co. and Kodansha Ltd., Amsterdam, Oxford, New York, Tokyo, 1981.

[4] Itô, K., McKean, H.P.: Diffusion processes and their sample paths. Springer Verlag, Berlin, Heidelberg, New York, 1974.

[5] Knight, F.: Random walks and a sojourn density process of Brownian motion. Trans. Amer. Math. Soc. 109, 56-86 (1963).

[6] Lévy, P.: Processus stochastiques et mouvement brownien. Gautier-Villars, Paris, 1948.

[7] Lévy, P.: Sur certains processus stochastiques homogénes. Compositio Math. 7, 283-339 (1939)

[8] McKean, H.P.: Brownian local time. Adv. Math. 15, 91-111 (1975).

[9] Perkins E.: The exact Hausdorff measure of the level sets of Brownian motion. Z. Wahrscheinlichkeitstheorie verw. Gebiete 58, 373-388 (1981)

[10] Ray, D.B.: Sojourn times of a diffusion process. Ill. J. Math. 7, 615-630 (1963).

[11] Taylor, S.J., Wendel,J.G.: The exact Hausdorff measure of the zero set of a stable process. Z. Wahrscheinlichkeitstheorie verw. Gebiete 6, 170-180 (1966)

[12] Trotter, H.F.: A property of Brownian motion paths. Ill. J. Math. 2, 425-433 (1958).

Paavo Salminen