On occupation time identity for reflecting Brownian motion with drift

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Abstract

This note is about an occupation time identity derived in [14] for reflecting Brownian motion with drift $-\mu < 0$, RBM($-\mu$), for short. The identity says that for RBM($-\mu$) in stationary state

$$(I_t^+, I_t^-) \stackrel{(d)}{=} (t - G_t, D_t - t), \qquad t \in \mathbb{R},$$

where G_t and D_t denote the starting time and the ending time, respectively, of an excursion from 0 to 0 (straddling t) and I_t^+ and I_t^- are the occupation times above and below, respectively, of the observed level at time t during the excursion. Due to stationarity, the common distribution does not depend on t. In fact, it is proved in [9] that the identity is true, under some assumptions, for all recurrent diffusions and stationary processes. In the null recurrent diffusion case the common distribution is not, of course, a probability distribution. The aim of this note is to increase understanding of the identity by studying the RBM($-\mu$) case via Ray-Knight theorems.

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1 Introduction

Consider a reflecting Brownian motion taking values on $\mathbb{R}_+$ with drift $-\mu < 0$. As is well known, this process is stationary having the exponential distribution with parameter 2μ as its stationary probability distribution. We take the whole of $\mathbb{R}$ to be the time axis and let $X = \{X_t : t \in \mathbb{R}\}$ denote this process. The symbols $\mathbf{E}$ and $\mathbf{P}$ refer to the expectation operator and the probability measure, respectively, associated with X. To simplify the notation we consider excursions of X straddling 0 instead of general t (cf. Abstract), and define

$$G := \sup\{s \leq 0 : X_s = 0\}, \quad D := \inf\{s > 0 : X_s = 0\},$$

and

$$I^+ := \int_G^D \mathbf{1}_{\{X_s > X_0\}}\, ds, \quad I^- := \int_G^D \mathbf{1}_{\{X_s < X_0\}}\, ds.$$

Then from [14] and [9] we have the following result

Theorem 1.1. *The random variables I^+, I^-, D and $-G$ are identical in law,*

$$(I^+, I^-) \overset{(d)}{=} (-G, D), \tag{1.1}$$

and conditionally on $I^+ + I^- = l$ the random variabel I^+ is uniformly distributed on $(0, l)$. Moreover,

$$\mathbf{E}\left(\exp\left(-\alpha I^+ - \beta I^-\right)\right) = \frac{2\mu}{\sqrt{2\alpha + \mu^2} + \sqrt{2\beta + \mu^2}} \tag{1.2}$$

and

$$\mathbf{P}\left(I^+ \in dt, I^- \in ds\right) = \frac{\mu}{\sqrt{2\pi(t+s)^3}} \; \exp\left(-\frac{\mu^2}{2}(t+s)\right) ds\, dt. \tag{1.3}$$

Our interest in RBM($-\mu$) arises from queueing theory where RBM($-\mu$) is obtained as a heavy traffic limiting process. However, as shown in [9], (1.1) is valid under some assumptions in much more generality – for all recurrent diffusions and stationary processes. We remark that the appearance of the uniform distribution after conditioning is a general property true for all cyclically stationary processes, see Kallenberg [8]. This is discussed in detail in [9].

For null recurrent diffusions the associated distributions are not probability distributions. As an example, consider the case with Brownian motion. Let $\{B_t^{(1)} : t \geq 0\}$ and $\{B_t^{(2)} : t \geq 0\}$ be two independent standard Brownian motions such that $B_0^{(1)} = B_0^{(2)} = x > 0$, and define for $t \in \mathbb{R}$

$$X_t^\circ := \begin{cases} B_t^{(1)}, & t \geq 0, \\ B_{-t}^{(2)}, & t \leq 0. \end{cases} \tag{1.4}$$

We randomize $X_0^\circ = x$ by distributing x on $\mathbb{R}_+$ according to $2\,dx$. Defining I^+, I^-, D and G as above it is a simple matter to verify, using, e.g., [4] formula 1.2.6.1 p. 202, that

$$\begin{aligned} \int_0^\infty 2dx\,\mathbf{E}_x\left(\exp\left(-\alpha I^+ - \beta I^-\right)\right) &= \int_0^\infty 2dx\,\mathbf{E}_x\left(\exp\left(-\alpha D + \beta G\right)\right) \\ &= \frac{\sqrt{2}}{\sqrt{\alpha} + \sqrt{\beta}} \end{aligned} \tag{1.5}$$

and

$$\int_0^\infty 2dx\,\mathbf{P}_x\left(I^+ \in dt, I^- \in ds\right) = \frac{1}{\sqrt{2\pi(t+s)^3}}\,ds\,dt, \tag{1.6}$$

where $\mathbf{P}_x$ refers to the conditional probability law of X° given that $X_0^\circ = x$. Notice, however, that for one-sided functionals we have

$$\int_0^\infty 2dx\,\mathbf{E}_x\left(\exp\left(-\alpha\int_0^D \mathbf{1}_{\{X_s^\circ > x\}}ds\right)\right) = \infty.$$

The law of the process $\{X_s^\circ : G < s < D\}$ is closely connected to the Itô's excursion law of BM, for this see Bismut [3] and Pitman [10].

Because X is a linear diffusion it is reversable in time, that is, $\{X_t\}$ and $\{X_{-t}\}$ are identical in law. Consequently, D and $-G$ are identical in law. The fact that I^+ and I^- are identical in law might seem surprising due to unsymmetry of the functionals. The first main issue of the paper, taken up in the the next section, is to show that also this fact can be seen, via Ray-Knight theorems, as a consequence of reversibility - but now in the space variable. This observation allows us to extend the result for some integral functionals (see Theorem 2.5). The second main issue treated in the third section of the paper is to understand more deeply why I^+ and D, say, are identical in law. This is achieved by connecting in the Ray-Knight setting these functionals to each other via random time change. For this method, a number of examples, and further references, see [15].

2 Reversibility in space

It is easy to show, using [4] formula 2.2.6.1. p. 300, that I^+ and I^- are identical in law. We now explain and extend this fact by showing that the total local time process in space variable associated with excursions straddling 0 is reversible.

Let $\{L(t,u) : t \geq 0, y \geq 0\}$ denote the local time of $\{X_s : s \geq 0\}$ up to time t at level u with respect to the Lebesgue measure, i.e., it holds a.s. that

$$\int_0^t g(X_s)\,ds = \int_0^\infty g(u)L(t,u)\,du \tag{2.1}$$

for any non-negative Borel function g. Our starting point is the Ray–Knight theorem (see [4] pp. 90–91) which describes the behaviour of the local time process L up to D given $X_0 = x$. To state this, let n be a non-negative integer and let $Z^{(n,2\mu)}$ denote the squared radial n-dimensional Ornstein–Uhlenbeck process with parameter μ. Recall that the generator of $Z^{(n,2\mu)}$ is

$$2z\frac{d^2}{dz^2} + (n - 2\mu z)\frac{d}{dz}.$$

The processes with $n = 0$ and $n = 4$ are important in the sequel. Notice that $Z^{(0,2\mu)}$ hits 0 a.s. where it is trapped (i.e., 0 is an exit-non-entrance boundary point). On the other hand, $Z^{(4,2\mu)}$ is positively recurrent and for this process 0 is an entrance-non-exit point.

Theorem 2.1. *Conditionally on $X_0 = x$,*

$$\{L(D,u) : 0 \leq u \leq x\} \stackrel{(d)}{=} \{Z_u^{(2,2\mu)} : 0 \leq u \leq x\},$$

$$\{L(D,x+u) : u \geq 0\} \stackrel{(d)}{=} \{Z_u^{(0,2\mu)} : u \geq 0\},$$

where the process $Z^{(0,2\mu)}$ is started from the position of $Z^{(2,2\mu)}$ at time x but otherwise $Z^{(2,2\mu)}$ and $Z^{(0,2\mu)}$ are independent.

It is convenient to assume that X is constructed from two independent reflecting Brownian motions with drift $-\mu$, denoted by $X^{(1)}$ and $X^{(2)}$, as explained above for a standard BM (cf. (1.4)). Let

$$D^{(1)} := \inf\{s \geq 0 : X_s^{(1)} = 0\}, \quad D^{(2)} := \inf\{s \geq 0 : X_s^{(2)} = 0\},$$

and define the total local time process over an excursion straddling 0 by

$$L^{(e)}(u) := L^{(1)}(D^{(1)}, u) + L^{(2)}(D^{(2)}, u), \quad u \geq 0,$$

where $L^{(1)}$ and $L^{(2)}$ are the local time processes associated with $X^{(1)}$ and $X^{(2)}$, respectively. Since $X^{(1)}$ and $X^{(2)}$ are independent given $X_0 = x$, it follows that also $L^{(1)}$ and $L^{(2)}$ are independent given $X_0 = x$.

Theorem 2.2. *The law of the total local time process $L^{(e)}$ is given by*

$$\{L^{(e)}(u) : 0 \leq u \leq X_0\} \stackrel{(d)}{=} \{Z_u^{(4,2\mu)} : 0 \leq u \leq T\},$$

$$\{L^{(e)}(X_0 + u) : u \geq 0\} \stackrel{(d)}{=} \{Z_u^{(0,2\mu)} : u \geq 0\},$$

where $Z^{(4,2\mu)}$ is started from 0, T is an exponentially distributed random variable with mean $1/(2\mu)$ independent of $Z^{(4,2\mu)}$, and $Z^{(0,2\mu)}$ is started from the position of $Z^{(4,2\mu)}$ at time T but otherwise $Z^{(4,2\mu)}$ and $Z^{(0,2\mu)}$ are independent. Moreover, $L^{(e)}(X_0)$ is exponentially distributed with mean $1/\mu$.

Proof. Recalling that X_0 is an exponentially distributed random variable with mean $1/(2\mu)$ the claim follows immediately from Theorem 2.1 since by Shiga and Watanabe [16] (see also [13] p. 440 and 448, and [4] p. 72)

$$\{Z_u^{(n,2\mu)} + Z_u^{(m,2\mu)} \ : \ u \geq 0\} \stackrel{(d)}{=} \{Z_u^{(n+m,2\mu)} \ : \ u \geq 0\},$$

where on the left hand side $Z^{(n,2\mu)}$ and $Z^{(m,2\mu)}$ are assumed to be independent, and the initial value of $Z^{(n+m,2\mu)}$ is taken to be the sum of the inital values of $Z^{(n,2\mu)}$ and $Z^{(m,2\mu)}$. The distribution of $L^{(e)}(X_0)$ can be computed in various ways; here it is perhaps instructive to use the result just derived. Hence, consider

$$\mathbf{P}\left(L^{(e)}(X_0) \in dz\right) = \mathbf{P}\left(Z_T^{(4,2\mu)} \in dz\right) = 2\mu\, G_{2\mu}^{(4)}(0, z)\, m^{(4)}(dz),$$

where $G^{(4)}$ is the symmetric Green kernel and $m^{(4)}$ is the speed measure associated with $Z^{(4,2\mu)}$. Analyzing the explicit expressions in [4] p. 141 and 641 it is seen that $G_{2\mu}^{(4)}(0, z) = 1/z$, and, since $m^{(4)}(dz) = \frac{1}{2}\, z \exp(-\mu z)\, dz$, the claim is proved. □

Next we show that the processes $Z^{(4,2\mu)}$ and $Z^{(2,2\mu)}$, as introduced in the Theorem 2.2, are time reversals of each other. This property may be deduced also by comparing the Ray-Knight theorems (A) and (B) in No. 11 p. 90 in [4]. To make the presentation more readable we give a direct proof of this key result.

Proposition 2.3. *Suppose that $Z_0^{(0,2\mu)}$ is exponentially distributed with mean $1/\mu$. Let $\zeta := \inf\{u : Z_u^{(0,2\mu)} = 0\}$. Then*

$$\{Z_{\zeta-u}^{(0,2\mu)} : 0 \le u \le \zeta\} \overset{(d)}{=} \{Z_u^{(4,2\mu)} : 0 \le u \le T\}, \tag{2.2}$$

where $Z^{(4,2\mu)}$ is started from 0 and T is an exponentially distributed random variable with mean $1/(2\mu)$ independent of $Z^{(4,2\mu)}$.

Proof. The claim (2.2) is equivalent with the following:

$$\begin{aligned}\mathbf{P}(Z_0^{(0,2\mu)} \in dz,\, Z_{\zeta-u_1}^{(0,2\mu)} \in dy_1,\, \dots, Z_{\zeta-u_n}^{(0,2\mu)} \in dy_n,\, \zeta \in dv) \\ = \mathbf{P}(Z_{u_1}^{(4,2\mu)} \in dy_1,\, \dots, Z_{u_n}^{(4,2\mu)} \in dy_n,\, Z_T^{(4,2\mu)} \in dz,\, T \in dv),\end{aligned} \tag{2.3}$$

where $0 < u_1 < u_2 < \cdots < u_n < v$. The identity (2.3) is proved using the explicit expressions (see [4] pp. 140-142) for the transition densities for $Z^{(0,2\mu)}$ and $Z^{(4,2\mu)}$. We have

$$\mathbf{P}_x(Z_t^{(4,2\mu)} \in dy) = p^{(4)}(t; x, y)\, m^{(4)}(dy)$$

with

$$p^{(4)}(t; x, y) := \frac{\mu \mathrm{e}^{2\mu t}}{\sqrt{xy}\,\sinh(\mu t)} \exp\left(-\frac{\mu \mathrm{e}^{-\mu t}(x+y)}{2\sinh(\mu t)}\right) I_1\left(\frac{\mu\sqrt{xy}}{\sinh(\mu t)}\right)$$

and

$$m^{(4)}(dy) := \frac{1}{2}\, y\, \mathrm{e}^{-\mu y}\, dy.$$

For $Z^{(0,2\mu)}$ it holds

$$\mathbf{P}_x(Z_t^{(0,2\mu)} \in dy) = p^{(0)}(t; x, y)\, m^{(0)}(dy)$$

with

$$p^{(0)}(t; x, y) = xy\mathrm{e}^{-2\mu t}\, p^{(4)}(t; x, y) \tag{2.4}$$

and

$$m^{(0)}(dy) = y^{-2}\, m^{(4)}(dy). \tag{2.5}$$

The $\mathbf{P}_x$-distribution of ζ can be computed by differentiating $p^{(0)}(t; x, y)$ with respect to y and letting $y \to 0$ (cf. [6] p. 154). This yields

$$\begin{aligned} n_x^{(0)}(0, t) := \mathbf{P}_x(\zeta \in dt)/dt &= \frac{\mu^2\, x}{2\,\sinh^2(\mu t)} \exp\left(-\frac{\mu \mathrm{e}^{-\mu t}\, x}{2\sinh(\mu t)}\right) \\ &= x\, \mathrm{e}^{-2\mu t} p^{(4)}(t; 0, x). \end{aligned} \tag{2.6}$$

The left hand side of (2.3) can be written as

$$
\begin{aligned}
\mathbf{P}(Z_0^{(0,2\mu)} \in dz,\, Z_{\zeta-u_1}^{(0,2\mu)} \in dy_1,\, \dots,\, Z_{\zeta-u_n}^{(0,2\mu)} \in dy_n,\, \zeta \in dv) \\
= \mu\, \mathrm{e}^{-\mu z}\, dz\; p^{(0)}(v-u_n; z, y_n)\, m^{(0)}(dy_n) \times \dots \\
\times p^{(0)}(u_2-u_1; y_2, y_1)\, m^{(0)}(dy_1)\; n_{y_1}^{(0)}(0,u_1)\, dv.
\end{aligned}
$$

Using (2.4), (2.5) and (2.6) it is seen that this equals the right hand side of (2.3), as claimed. □

The notion of reversibility in space is made precise in the next theorem. The proof of the theorem is an immediate consequence of Theorem 2.2 and Proposition 2.3.

Theorem 2.4. *The total local time process $\{L^{(e)}(u) : u \geq 0\}$ is reversible, i.e., letting $\zeta(L) := \inf\{u > 0 : L^{(e)}(u) = 0\}$ it holds*

$$
\{L^{(e)}(u) : 0 \leq u \leq \zeta(L)\} \overset{(d)}{=} \{L^{(e)}(\zeta(L) - u) : 0 \leq u \leq \zeta(L)\}.
$$

Moreover,

$$
\{L^{(e)}(X_0 + u) : u \geq 0\} \overset{(d)}{=} \{L^{(e)}(X_0 - u) : 0 \leq u \leq X_0\} \overset{(d)}{=} \{Z_u^{(0,2\mu)} : u \geq 0\},
$$

where $Z_0^{(0,2\mu)}$ is exponentially distributed with mean $1/\mu$.

Reversibility of the total local time process implies clearly the fact that I^+ and I^- are identical in law and also leads to the following generalization

Theorem 2.5. *Let g be an even Borel measurable function bounded from below. Then*

$$
\int_G^D g(X_s - X_0)\, \mathbf{1}_{\{X_s > X_0\}}\, ds \overset{(d)}{=} \int_G^D g(X_s - X_0)\, \mathbf{1}_{\{X_s < X_0\}}\, ds. \tag{2.7}
$$

Proof. By the occupation time formula and Theorem 2.4 we have, since $g(x) = g(-x)$,

$$
\begin{aligned}
\int_G^D g(X_s - X_0)\, \mathbf{1}_{\{X_s > X_0\}}\, ds \\
&= \int_0^\infty g(x)\, L^{(e)}(X_0 + x)\, dx \\
&\overset{(d)}{=} \int_0^\infty g(-x)\, L^{(e)}(X_0 - x)\, dx \\
&= \int_G^D g(X_s - X_0)\, \mathbf{1}_{\{X_s < X_0\}}\, ds,
\end{aligned}
$$

where the equalities in the first and the last step hold a.s. □

Remark 2.6. **1.** For the process X° (cf. (1.4)) constructed from Brownian motions we have similar results as above but expressed in terms of the squared Bessel diffusions $Z^{(n,0)}$. In particular, since X_0° is distributed on $\mathbb{R}_+$ according to $2\,dx$, it is seen that $L^{(e)}(X^\circ)$ and $Z_0^{(2,0)}$ are distributed according to dx. With these modifications and taking $\mu = 0$, the results in Proposition 2.3, Theorem 2.4 and Theorem 2.5 are valid.
2. For somewhat related work, we refer to Howard and Zumbrun [5], see also Pitman [11], where it is shown that occupation times for Brownian bridge are invariant under translations and reflections when the occupation set is in the interval between the initial and final value value of the bridge. For a Ray-Knight theorem for Brownian bridge see Jeulin [7], and for standard Brownian excursion bridge see Biane [1], Biane and Yor [2], and Pitman [12].

3 Random time change

In this section we discuss the identity in law between I^+ and D (recall from Theorem 1.1 that $I^+ \stackrel{(d)}{=} I^-$ and $D \stackrel{(d)}{=} -G$). As pointed out in the Introduction, it is a simple matter to prove the identity by straightforward computations. Hence, the aim of the treatment here is to bring in some features behind the identity. The following discussion utilizies the approach in [15]; see, in particular, the first proof of the Biane-Imhof identity.

Let $\{W_t\}$ denote a standard Brownian motion. From Theorem 2.4,

$$I^{(+)} \stackrel{(d)}{=} \int_0^\zeta Z_s\,ds,$$

where the process $Z := Z^{(0,2\mu)}$ is the solution of the SDE

$$dZ_t = 2\sqrt{Z_t}\,dW_t - 2\mu\,Z_t\,dt \tag{3.1}$$

with Z_0 exponentially distributed with mean $1/\mu$, and $\zeta := \inf\{t : Z_t = 0\}$. Assume, for a moment, that Z_0 is deterministic and equal to x, say. Introduce

$$A_t := \int_0^t Z_s\,ds$$

and observe that $A_t < \infty$ a.s. Let $\{\alpha_t\}$ denote the right-continuous inverse of $\{A_t\}$. Since the quadratic variation of the local martingale

$$Y_t := \int_0^t \sqrt{Z_s}\, dW_s$$

is given by

$$\langle Y, Y\rangle_t = \int_0^t Z_s\, ds = A_t,$$

it follows from Lévy's characterization theorem that $\{B_t\} := \{Y_{\alpha_t}\}$ is a Brownian motion started from zero and stopped at A_ζ. Hence, for $t < A_\zeta$

$$\begin{aligned} Z_{\alpha_t} - x &= 2\int_0^{\alpha_t} \sqrt{Z_s}\, dW_s - 2\mu \int_0^{\alpha_t} Z_s\, ds \\ &= 2Y_{\alpha_t} - 2\mu A_{\alpha_t} \\ &= 2B_t - 2\mu t. \end{aligned}$$

Letting $t \to A_\zeta$ implies $Z_{\alpha_t} \to 0$ and, consequently, $B^{(\mu)}_{A_\zeta} = x/2$, where $B^{(\mu)}_t := -B_t + \mu t$ is a BM with drift μ. Thus, taking into account that

$$0 < Z_{\alpha_t} = x + 2B_t - 2\mu t, \quad 0 \le t < A_\zeta,$$

we obtain

$$A_\zeta = \inf\{t : B^{(\mu)}_t = x/2\}. \tag{3.2}$$

Randomizing now Z_0 to be exponentially distributed with mean $1/\mu$ it follows that

$$I^+ \overset{(d)}{=} A_\zeta \overset{(d)}{=} H_\lambda(B^{(\mu)}) := \inf\{t : B^{(\mu)}_t = \lambda\}, \tag{3.3}$$

where λ is independent of $B^{(\mu)}$ and exponentially distributed with mean $1/(2\mu)$. Noting that, by spatial homogeneity,

$$H_\lambda(B^{(\mu)}) \overset{(d)}{=} D$$

we have proved the claim that $I^{(+)}$ and D are identical in law.

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