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# Brownian motion

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## 1 Introduction

Brownian motion, also called Wiener process, is probably the most important stochastic process. Its key position is due to various reasons. The aim of this introduction is to present shortly some of these reasons without striving anyway to a complete presentation. The reader will find Brownian motion appearing also in many other articles in this book, and can in this way form his/her own understanding of the importance of the matter.

Our starting point is that Brownian motion has a deep physical meaning being a mathematical model for movement of a small particle suspended in water (or some other liquid). The movement is caused by ever moving water molecules which constantly hit the particle. This phenomenon was observed already in the eighteenth century but it was Robert Brown (1773–1858), a Scottish botanist, who studied the phenomenon systematically and understood its complex nature. Brown was not, however, able to explain the reason for Brownian motion. This was done by Albert Einstein (1879–1955) in one of his famous papers in 1905. Einstein was not aware of the works of Brown but predicted on theoretical grounds that Brownian motion should exist. We refer to Nelson [10] for further readings on Brown's and Einstein's contributions.

However, before Einstein, Louis Bachelier (1870–1946), a French mathematician, studied Brownian motion from a mathematical point of view. Also Bachelier did not know Brown's achievements and was, in fact, interested in modeling fluctuations in stock prices. In his thesis *Théorie de la Spéculation* published in 1900 Bachelier used simple symmetric random walk as the first step to model price fluctuations. He was able to find the right normalisation for the random walk to obtain Brownian motion after a limiting procedure. In this way he found among other things that the location of the Brownian particle at a given time is normally distributed and that Brownian motion has independent increments. He also computed the distribution of the maximum of Brownian motion before a given time and understood the connection between Brownian motion and the heat equation. Bachelier's work has received much attention during the recent years and Bachelier is now considered to be the father of financial mathematics. See Cootner [4] for a translation of Bachelier's thesis.

The usage of the term Wiener process as a synonym for Brownian motion is to honor the work of Norbert Wiener (1894–1964). In the paper [13] Wiener constructs a probability space which carries a Brownian motion and thus proves the existence of Brownian motion as a mathematical object as given in the following

**Definition.** Standard one-dimensional Brownian motion initiated at $x$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is a stochastic process $W = \{W_t : t \geq 0\}$ such that

(a) $W_0 = x$ a.s.,

(b) $s \mapsto W_s$ is continuous a.s.,

(c) for all $0 = t_0 < t_1 < \ldots < t_n$ the increments

$$W_{t_n} - W_{t_{n-1}}, W_{t_{n-1}} - W_{t_{n-2}}, \ldots, W_{t_1} - W_{t_0}$$

are normally distributed with

$$\mathbf{E}(W_{t_i} - W_{t_{i-1}}) = 0, \qquad \mathbf{E}(W_{t_i} - W_{t_{i-1}})^2 = t_i - t_{i-1}.$$

Standard $d$-dimensional Brownian motion is defined as $\bar{W} = \{(W_t^{(1)}, \ldots, W_t^{(d)}) : t \geq 0\}$, where $W^{(i)}$, $i = 1, 2, \ldots, d$, are independent, one-dimensional standard Brownian motions.

Notice that, because uncorrelated normally distributed random variables are independent it follows from (c) that the increments of Brownian motion are independent. For different constructions of Brownian motion and also for Paul Lévy's (1886–1971) contribution for the early development of Brownian motion, see Knight [9].

Another reason for the central role played by Brownian motion is that it has many "faces". Indeed, Brownian motion is

- a strong Markov process,
- a diffusion,
- a continuous martingale,
- a process with independent and stationary increments,
- a Gaussian process.

The theory of stochastic integration and stochastic differential equations is a powerful tool to analyse stochastic processes. This so called stochastic calculus was first developed with Brownian motion as the integrator. One of the main aims hereby is to construct and express other diffusions and processes in terms of Brownian motion. Another method to generate new diffusions from Brownian motion is via random time change and scale transformation. This is based on the theory of local time which theory was initiated and much developed by Lévy.

We remark also that the theory of Brownian motion has close connections with other fields of mathematics like potential theory and harmonic analysis. Moreover, Brownian motion is an important concept in statistics; for instance, in the theory of the Kolmogorov–Smirnov statistic which is used to test the parent distribution of a sample.

Brownian motion is a main ingredient in many stochastic models. Many queueing models in heavy traffic lead to Brownian motion or processes closely related to it, see e.g. Harrison [6] and Prabhu [11]. Finally, in the famous Black–Scholes market model the stock price process $P = \{P_t \,:\, t \geq 0\}$, $P_0 = p_o$ is taken to be a geometric Brownian motion, that is,

$$P_t = p_o \, \mathrm{e}^{\sigma W_t + \mu t}.$$

Using Itô's formula it is seen that $P$ satisfies the stochastic differential equation

$$\frac{dP_t}{P_t} = \sigma\, dW_t + (\mu + \frac{\sigma^2}{2})\, dt.$$

In the next section we present basic distributional properties of Brownian motion. We concentrate on the one-dimensional case but it is clear that many of the these properties hold in general. The third section treats local properties of Brownian paths and in the last section we discuss the important Feynman–Kac formula for computing distributions of functionals of Brownian motion. For further details, extensions and proofs we refer to Freedman [5], Karatzas and Shreve [8], Ito and McKean [7], Knight [9], Revuz and Yor [12] and Borodin and Salminen [2]. For the Feynman–Kac formula, see also Borodin and Ibragimov [1] and Durrett [3].

# 2 Basic properties of Brownian motion

**Strong Markov property.** In the introduction above it is already stated that Brownian motion is a strong Markov process. To explain this more in detail let $\{\mathcal{F}_t \ : \ t \geq 0\}$ be the natural completed filtration of Brownian motion. Let $T$ be a stopping time with respect to this filtration and introduce the $\sigma$-algebra $\mathcal{F}_T$ of events occuring before the time point $T$, that is,

$$A \in \mathcal{F}_T \quad \Leftrightarrow A \in \mathcal{F} \text{ and } A \cap \{T \leq t\} \in \mathcal{F}_t.$$

Then the strong Markov property says that a.s. on the set $\{T < \infty\}$

$$\mathbf{E}(f(W_{t+T}) \,|\, \mathcal{F}_T) = \mathbf{E}_{W_T}(f(W_t)),$$

where $\mathbf{E}_x$ is the expectation operator of $W$ when started at $x$ and $f$ is a bounded and measurable function. The strong Markov property of Brownian motion is a consequence of the independence of increments.

**Spatial homogeneity.** Assume that $W_0 = 0$. Then for every $x \in \mathbf{R}$ the process $x + W$ is a Brownian motion initiated at $x$.

**Symmetry.** $-W$ is a Brownian motion initiated at 0 if $W_0 = 0$.

**Reflection principle.** Let $H_a := \inf\{t \ : \ W_t = a\}$, $a \neq 0$, be the first hitting time of $a$. Then the process given by

$$Y_t := \begin{cases} W_t, & t \leq H_a, \\ 2\,a - W_t, & t \geq H_a, \end{cases}$$

is a Brownian motion. Using reflection principle for $a > 0$ we can find the law of the maximum of Brownian motion before a given time $t$ :

$$\mathbf{P}_0(\sup\{W_s \ : \ s \leq t\} \geq a) = 2\,\mathbf{P}_0(W_t \geq a) = \frac{2}{\sqrt{2\pi\, t}} \int_a^\infty \mathrm{e}^{-x^2/2t} dx.$$

Further, because

$$\mathbf{P}_0(\sup\{W_s \ : \ s \leq t\} \geq a) = \mathbf{P}_0(H_a \leq t),$$

we obtain, by differentiating with respect to $t$, the density of the distribution of the first hitting time $H_a$ :

$$\mathbf{P}_0(H_a \in dt) = \frac{a}{\sqrt{2\pi\, t^3}}\ \mathrm{e}^{-a^2/2t} dt.$$

The Laplace transform of the distribution of $H_a$ is

$$\mathbf{E}_0\left(\mathrm{e}^{-\alpha H_a}\right) = \mathrm{e}^{-a\sqrt{2\alpha}}, \quad \alpha > 0.$$

**Reflecting Brownian motion.** The process $\{|W_t| : \ t \geq 0\}$ is called reflecting Brownian motion. It is a time-homogeneous, strong Markov process. A famous result due to Paul Lévy is that the processes $\{|W_t| : \ t \geq 0\}$ and $\{\sup\{W_s \ : \ s \leq t\} - W_t : \ t \geq 0\}$ are identical in law.

**Scaling.** For every $c > 0$ the process $\{\sqrt{c}\, W_{t/c} : \ t \geq 0\}$ is a Brownian motion.

**Time inversion.** The process given by

$$Z_t := \begin{cases} 0, & t = 0, \\ t\, W_{1/t}, & t > 0, \end{cases}$$

is a Brownian motion.

**Time reversibility.** Assume that $W_0 = 0$. Then for a given $t > 0$, the processes $\{W_s : \ 0 \leq s \leq t\}$ and $\{W_{t-s} - W_t : \ 0 \leq s \leq t\}$ are identical in law.

**Last exit time.** For a given $t > 0$ assuming that $W_0 = 0$ the last exit time of 0 before time $t$

$$\lambda_0(t) := \sup\{s \leq t : \ W_s = 0\}$$

is arcsine-distributed on $(0,t)$, that is,

$$\mathbf{P}_0(\lambda_0(t)\in dv)=\frac{dv}{\pi\sqrt{v(t-v)}}.$$

**Lévy's martingale characterization.** A continuous real-valued process $X$ in a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbf{P})$ is an $\mathcal{F}_t$-Brownian motion if and only if both $X$ itself and $\{X_t^2-t:\ t\ge 0\}$ are $\mathcal{F}_t$-martingales.

**Strong law of large numbers.**

$$\lim_{t\to\infty}\frac{W_t}{t}=0\quad\text{a.s.}$$

**Laws of the iterated logarithm**

$$\limsup_{t\downarrow 0}\frac{W_t}{\sqrt{2t\ln\ln(1/t)}}=1\quad\text{a.s.}$$

$$\limsup_{t\to\infty}\frac{W_t}{\sqrt{2t\ln\ln t}}=1\quad\text{a.s.}$$

# 3 Local properties of Brownian paths

A (very) nice property Brownian paths $t\mapsto W_t$ is that they are continuous (as already stated in the definition above). However, the paths are nowhere differentiable and the local maximum points are dense. Inspite of these irregularities, we are faced with astonishing regularity when learning that the quadratic variation of $t\mapsto W_t,\ t\le T$ is a.s. equal to $T$. Below we discuss in more details these and some other properties of Brownian paths.

**Hölder continuity and nowhere differentiability.** Brownian paths are a.s. locally Hölder continuous of order $\alpha$ for every $\alpha<1/2$. In other words, for all $T>0,\ 0<\alpha<1/2$ and almost all $\omega$ there exists a constant $C_{T,\alpha}(\omega)$ such that for all $t,s<T$,

$$|W_t(\omega)-W_s(\omega)|\le C_{T,\alpha}(\omega)|t-s|^{\alpha}.$$

Brownian paths are a.s. nowhere locally Hölder continuous of order $\alpha\ge 1/2$. In particular, Brownian paths are nowhere differentiable.

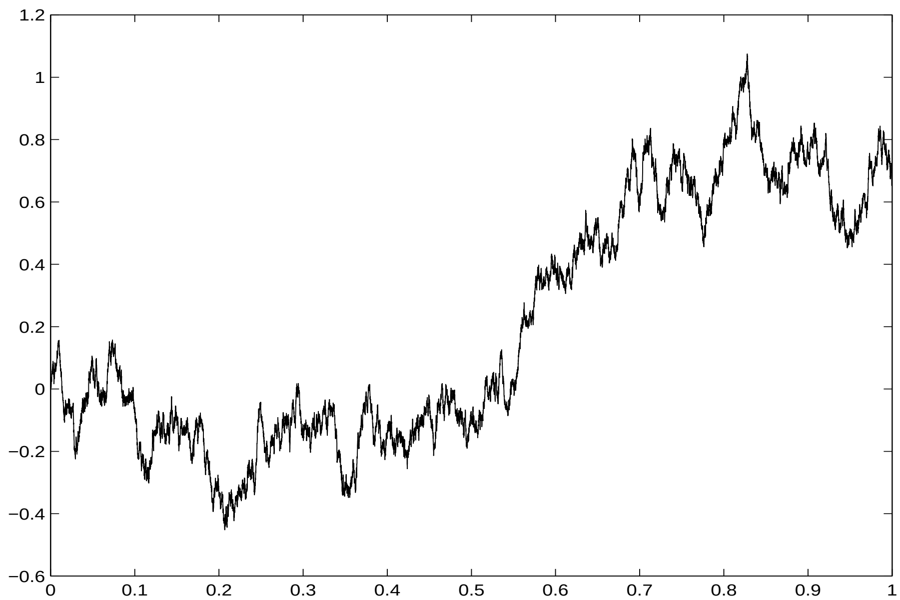


Figure 1: A realization of a standard Brownian motion (by Margret Halldorsdottir).

**Lévy's modulus of continuity.**

$$\limsup_{\delta\to 0}\sup\frac{|W_{t_2}-W_{t_1}|}{\sqrt{2\delta\ln(1/\delta)}}=1 \quad \text{a.s.}$$

where the supremum is over $t_1$ and $t_2$ such that $|t_1 - t_2| < \delta$.

**Variation.** Brownian paths are of infinite variation on intervals, that is, for every $s \le t$ a.s.

$$\sup\sum|W_{t_i}-W_{t_{i-1}}|=\infty$$

where the supremum is over all subdivisions $s \le t_1 \le \ldots \le t_n \le t$ of the interval $(s,t)$. On the other hand, let $\Delta_n := \{t_i^{(n)}\}$ be a sequence of subdivisions of $(s,t)$ such that $\Delta_n \subseteq \Delta_{n+1}$ and

$$\lim_{n\to\infty}\max_i |t_i^{(n)}-t_{i-1}^{(n)}|=0.$$

Then a.s. and in $\mathbf{L}^2$,

$$\lim_{n\to\infty}\sum|W_{t_i^{(n)}}-W_{t_{i-1}^{(n)}}|^2=t-s.$$

This fact is expressed by saying that the *quadratic variation* of $W$ over $(s,t)$ is $t-s$.

**Local maxima and minima.** Recall that for a continuous function $f:[0,\infty)\to\mathbf{R}$ a point $t$ is called a point of local (strict) maximum if there exists $\epsilon > 0$ such that for all $s \in (t-\epsilon, t+\epsilon)$ we have $f(s) \le f(t)$ ($f(s) < f(t)$, $s \ne t$). A point of local minimum is defined analogously. Then for almost every $\omega \in \Omega$ the set of points of local maxima for the Brownian path $W_\cdot(\omega)$ is countable and dense in $[0,\infty)$, each local maximum is strict, and there is no interval on which the path is monotone.

**Points of increase and decrease.** Recall that for a continuous function $f:[0,\infty)\to\mathbf{R}$ a point $t$ is called a point of increase if there exists $\epsilon > 0$ such that for all $s \in (0,\epsilon)$ we have $f(t-s) \le f(t) \le f(t+s)$. A point of decrease is defined analogously. Then for almost every $\omega \in \Omega$ the Brownian path $W_\cdot(\omega)$ has no points of increase or decrease.

**Level sets.** For a given $\omega$ and $a \in \mathbf{R}$ let $\mathcal{Z}_a(\omega) := \{t:\ W_t(\omega) = a\}$. Then a.s. the random set $\mathcal{Z}_a(\omega)$ is unbounded and of the Lebesgue measure 0. It is closed and has no isolated points, i.e., is dense in itself. A set with these properties is called *perfect*. The Hausdorff dimension of $\mathcal{Z}_a$ is $1/2$.

# 4 Feynman–Kac formula for Brownian motion

Consider the function

$$v(t,x) := \mathbf{E}_x\Big(F(W_t)\exp\Big(\int_0^t f(t-s,W_s)\,ds\Big)\Big),$$

where $f$ and $F$ are bounded and Hölder continuous ($f$ locally in $t$). Then the Feynman–Kac formula says that $v$ is the unique solution of the differential problem

$$\begin{aligned} u_t'(t,x) &= \frac{1}{2}\,u_{xx}''(t,x) + f(t,x)u(t,x), \\ u(0,x) &= F(x). \end{aligned}$$

Let now $\tau$ be an exponentially distributed (with parameter $\lambda$) random variable, and consider the function

$$v(x) := \mathbf{E}_x\Big(F(W_\tau)\exp\Big(-\gamma\int_0^\tau f(W_s)\,ds\Big)\Big),$$

where $\gamma \geq 0$, $F$ is piecewise continuous and bounded, $f$ is piecewise continuous and non-negative. Then $v$ is the unique bounded function with continuous derivative which on every interval of continuity of $f$ and $F$ satisfies the differential equation

$$\frac{1}{2}\,u''(x) - (\lambda + \gamma\, f(x))u(x) = -\lambda F(x).$$

We give some examples and refer to [2] for more:

**Arcsine law.** For $F \equiv 1$ and $f(x) = \mathbf{1}_{(0,\infty)}(x)$ we have

$$\mathbf{E}_0\Big(\exp\Big(-\gamma\int_0^\tau \mathbf{1}_{(0,\infty)}(W_s)\,ds\Big)\Big)$$

$$= \begin{cases} \frac{\lambda}{\lambda+\gamma} - \Big(\frac{\lambda}{\lambda+\gamma} - \frac{\sqrt{\lambda}}{\sqrt{\lambda+\gamma}}\Big)\mathrm{e}^{-x\sqrt{2\lambda+2\gamma}}, & \text{if } x \geq 0, \\ 1 - \Big(1 - \frac{\sqrt{\lambda}}{\sqrt{\lambda+\gamma}}\Big)\mathrm{e}^{x\sqrt{2\lambda+2\gamma}}, & \text{if } x \leq 0. \end{cases}$$

Inverting this double Laplace transform when $x = 0$ gives

$$\mathbf{P}_0\Big(\int_0^t \mathbf{1}_{(0,\infty)}(W_s)\,ds \in dv\Big) = \frac{dv}{\pi\sqrt{v(t-v)}}.$$

**Cameron–Martin formula.** For $F \equiv 1$ and $f(x) = x^2$ we have

$$\mathbf{E}_0\Big( \exp\big( - \gamma \int_0^t W_s^2\, ds\big)\Big) = (\cosh(\sqrt{2\gamma} t))^{-1/2}.$$

**An occupation time formula for Brownian motion with drift.** For $\mu > 0$

$$\mathbf{E}_0\Big(\exp\Big(-\gamma \int_0^\infty \mathbf{1}_{(-\infty,0)}(W_s + \mu\, s)\, ds\Big)\Big) = \frac{2\mu}{\sqrt{\mu^2 + 2\gamma} + \mu}.$$

**Dufresne's formula for geometric Brownian motion.** For $a > 0$ and $b > 0$

$$\mathbf{E}_0\Big(\exp\Big(-\gamma \int_0^\infty \mathrm{e}^{a\, W_s - b\, s}\, ds\Big)\Big) = \frac{2^{\nu+1}\gamma^\nu}{a^{2\nu}\Gamma(2\nu)} K_{2\nu}\Big(\frac{2\sqrt{2\gamma}}{a}\Big),$$

where $\nu = b/a^2$ and $K_{2\nu}$ is the modified Bessel function of second kind and of order $2\nu$. The Laplace transform can be inverted

$$\mathbf{P}_0\Big(\int_0^\infty \mathrm{e}^{a\, W_s - b\, s}\, ds \in dy\Big) = \frac{2^{2\nu}}{a^{4\nu}\Gamma(2\nu)} y^{-2\nu-1} \mathrm{e}^{-2/a^2 y}\, dy.$$

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