On busy periods of the unbounded Brownian storage

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Abstract

A stationary storage process with Brownian input and constant service rate is studied. Explicit formulae for quantities related to busy periods (excursions) are derived. In particular, we compute the distributions of the occupation times the process spends above and below, respectively, the present level during the on-going busy period, and make the surprising observation that these occupation times are identically distributed.

1 Introduction

Reflected Brownian motion (RBM) with negative drift entered queueing theory in 1960's in works of Kingman and others in the role of a general heavy traffic limit process. Later, Harrison [10], and references therein, and Abate and Whitt in a series of papers, see especially [1], [2], [3], [4], and [5]) started considering the process and its more complex relatives as mathematical models of their own right, interesting also without an explicit justification as heavy traffic limits of "real" queueing or storage systems. Moreover, much work has been done to analyze queueing networks and multiclass service stations with Brownian input; for this see Harrison and Williams [11] and references therein.

Although the literature on these processes is already quite extensive as regards solutions to difficult and sophisticated problems on queueing networks and various queueing disciplines, there seems to be no concise presentation of what is known in terms of explicit formulae about the very simplest case, a storage with unbounded buffer and Brownian input. In comparison to any other storage system (probably even $M/M/1$ included) this one has most, and often beautiful, explicitely known properties; it could be used more often as a prototype model for simple queues, even in textbooks.

We have not found very many results in the literature concerning especially on-going busy periods (excursions) of the Brownian storage in stationary state (see, however, the above cited papers by Abate and Whitt, where the remaining length of the on-going busy period, also called the busy period distribution, is carefully studied). The aim of this paper is to fill a part of this gap by presenting explicit expressions for some basic distributions related to on-going busy periods. Some of the results are rather interesting and surprising. For example, we show that during the on-going busy period, the occupation times the process spends above and below the present level are identically distributed! It is observed that this common distribution is also the distribution of the remaining length of the on-going busy period.

The paper is organized as follows. In the next section we define and discuss the concept of the stationary Brownian storage process on the time parameter set $\mathbb{R}$. In particular, we prove the reversibility of the storage process by exploiting symmetry of the transition density. Although the result is well-known we believe that the presented simple proof based on the symmetry of the transition density is not widely known. This symmetry property is also a key to a partial explanation of the equality in distribution of the considered occupation times (see Proposition 3.11). We also point out, as a concequence of reversibility, in Corollary 2.11 an interesting counterpart (in which the future infimum of the Brownian motion with drift is used) of the classical construction of RBM based on Skorohod's reflection equation (see Harrison [10]). Section 3 is devoted to the analysis of busy periods (excursions). In sections 3.1 and 3.2 we characterize the law of the stationary on-going busy period. In the first (obvious) characterization, a busy period is described via two conditionally independent and identically distributed killed Brownian motions with drift, the first one running forward and the second one backward in time, respectively. Although the first characterization is very practical from the point of view of computations, it is natural to explain – and this is done in section 3.2 – the structure of a busy period with (only) forward running time. In section 3.3 we present the results on occupation times and lenghts of busy periods advertised above. Finally, the distribution of the maximum within a busy period is determined in section 3.4. It seems to us that the results concerning the whole on-going busy period are in a sense simpler than the corresponding results on the remaining busy period.

2 Definitions and preliminaries

2.1 Storage process

Since we are mainly interested in the stationary storage process, we consider a Brownian motion in which the time parameter attains values on the whole real line. In a sense this is equivalent to considering two independent Brownian motions starting from 0, the first one running forward in time and the other one backward in time. However, to fix ideas we use the following

Definition 2.1 Brownian motion on the time axis $\mathbb{R}$ passing through 0 at time $u \in \mathbb{R}$ is a continuous Gaussian process $W^{(u)} = \{W_t^{(u)} \ : \ t \in \mathbb{R}\}$ in $\mathbb{R}$ such that for all t and s

(i) $W_u^{(u)} = 0$,

(ii) $\mathbf{E}\, W_t^{(u)} = 0$,

(iii) $\mathbf{E}\, W_t^{(u)} W_s^{(u)} = \frac{1}{2}(|t-u| + |s-u| - |t-s|)$.

The process $W^{(0)}$ is denoted by W.

We have the following basic result which is much used below. The proof is a straightforward computation and is left to the reader.

Proposition 2.2 For any $u \in \mathbb{R}$ the process $\{W_t^{(u)} - W_0^{(u)} \ : \ t \in \mathbb{R}\}$ is a BM passing through 0 at time 0.

Let us consider an unlimited storage (buffer) with one input and one output. Harrison [10] models the *net* input (netput) process in time interval

$[0, t]$, that is, the input to a buffer minus the *potential* output (the realized output will be less because the buffer is sometimes empty), as a Brownian motion with negative drift. To make our intuition a bit more fixed, let us here consider W as the input process and $t \mapsto \mu t$, with $\mu > 0$, as the potential output process, i.e. we consider a fluid queue with constant service rate. The main object of this paper is now defined in terms of W.

Definition 2.3 The stationary storage process $V = \{V_t : t \in \mathbb{R}\}$ with standard Brownian input, constant service rate μ, and unbounded buffer, the *Brownian storage,* is defined as

$$V_t := \sup_{-\infty < s \leq t} \{W_t - W_s - \mu(t - s)\}, \quad t \in \mathbb{R}, \ \mu > 0.$$

From proposition 2.2 it follows that for any $u \in \mathbb{R}$

$$V_t = \sup_{-\infty < s \leq t} \{W_t^{(u)} - W_s^{(u)} - \mu(t - s)\}. \tag{1}$$

Consequently, the distribution of V_t is independent of t. Combining this with the fact that W has stationary increments shows, as indicated in the definition, that V is stationary. In fact, choosing in (1) $t = u$ and using the well-known (see e.g. [6, p. 14 and 109]) result on the maximum of a Brownian motion with negative drift yield (see also (3) in section 2.2.1)

Proposition 2.4 The stationary distribution of V is exponential with parameter 2μ, i.e.,

$$\mathbf{P}(V_u > x) = e^{-2\mu x} \qquad \forall u \in \mathbb{R}.$$

This way to define the workload (or virtual waiting time) process as a simple supremum dates back to Reich [15]. The idea of the definition becomes intuitively understandable by noting that the latest time before t when the buffer is empty is a value s^* where the supremum is reached.

Remark 2.5 The definition is easily modified to give a non-stationary storage process that is zero for $t < 0$. Let U be a non-negative random variable independent of W. The storage process starting from level U can be defined as

$$V_t^U := \sup_{-\infty < s \leq t} \{A_t - A_s - \mu(t - s)\}, \quad t \geq 0,$$

where $A_s = -U \cdot \mathbf{1}_{(-\infty,0)}(s) + W_s \cdot \mathbf{1}_{[0,\infty)}(s)$.

2.2 Relation to reflecting Brownian motion

2.2.1 Preliminaries on reflecting Brownian motion

To make the paper self-contained we recall here some well known properties of reflected Brownian motion with negative drift. Much of this can be found in [10]. On the other hand, because the process is a diffusion, it can also be analyzed using the diffusion theory presented in Itô and McKean [12]. In particular, we want to stress the importance of existence of symmetric transition density (see (4), (5)) which is much used below, e.g., when discussing time reversibility. It seems to us that this fact has not been exploited when analyzing the simple model considered here. We also remark that the existence of symmetric density is a general property which all linear diffusions have. In [6, p. 110] some basic facts about reflected Brownian motion with drift are listed, and many distributions can be derived from the corresponding distributions of reflecting Brownian motion without drift, using absolute continuity.

One way to construct reflected Brownian motion with drift $-\mu$ ($\mu > 0$) is to use Skorokhod's reflection equation (see Rogers and Williams [17, p. 117] or [10, p. 19]). This generalizes the classical construction of reflected Brownian motion due to Lévy (see e.g. [12, p. 40]). Because Skorokhod's result is of key importance in our approach we recall it in the next

Theorem 2.6 The stochastic equation

$$X_t = W_t - \mu t + Y_t, \quad t \geq 0,$$

where X and Y are unknown has a (strong) unique solution such that

(i) $X_t \geq 0$, $t \mapsto Y_t$ is increasing, and $Y_0 = 0$,

(ii) $\int_0^t \mathbf{1}_{\{X_s > 0\}} dY_s = 0$.

The process Y is given by $Y_t = -\inf_{0 \leq s \leq t}\{W_s - \mu s\}$.

Moreover, it can be proved (see [10, p. 81]) that X is a strong Markov process, and so a diffusion. Let $\mathbf{P}_x$ denote the probability measure associated to X when started at x. In [10, p. 49-50] it is stated that the transition density of X

$$(t, x) \mapsto p(t; x, y) = \frac{\partial}{\partial y}\, \mathbf{P}_x(X_t \leq y)$$

solves the differential system

$$\begin{aligned} u'_t(t,x) &= \frac{1}{2}u''_{xx}(t,x) - \mu u'_x(t,x), \quad x>0, \\ u'_x(t,0+) &= 0. \end{aligned}$$

In view of the diffusion theory, it is therefore natural to call X a reflected Brownian motion with drift $-\mu$, RBM$(-\mu)$, for short.

Using the joint distribution of W_t and Y_t the transition probability is computed in [10, p. 49]:

$$\mathbf{P}_x(X_t > y) = \Phi\Big(\frac{-y+x-\mu t}{\sqrt{t}}\Big) + e^{-2\mu y}\Phi\Big(\frac{-y-x+\mu t}{\sqrt{t}}\Big), \tag{2}$$

where Φ is the standard normal distribution function. Letting here $t \to \infty$ (cf. [10, p. 94]) we have for all (non-negative) x and y

$$\mathbf{P}_x(X_t > y) \to e^{-2\mu y}, \tag{3}$$

and it is easy to check that the stationary probability distribution of X is exponential with parameter 2μ. Another approach to this matter is to use diffusion theory and deduce that $m(dx) = 2\mu e^{-2\mu x}dx$ can be taken to be the speed measure of X with respect to which the transition probability has a symmetric density $\tilde{p}$ (see [12, p. 149]):

$$\mathbf{P}_x(X_t \in dy) = \tilde{p}(t;x,y)\, m(dy) = \tilde{p}(t;y,x)\, m(dy). \tag{4}$$

Consequently, m is an invariant probability measure, in other words, the stationary distribution of X.

The Laplace transform of $\tilde{p}$ is computed in [6, p. 110] (with a slightly different normalization, though) to be

$$\begin{aligned} G_\beta(x,y) &= \int_0^\infty e^{-\beta t}\tilde{p}(t;x,y)dt \\ &= \frac{e^{-(\gamma-\mu)x}}{\mu(\gamma-\mu)}\Big(\frac{\gamma-\mu}{2\gamma}e^{(\gamma+\mu)y} + \frac{\gamma+\mu}{2\gamma}e^{-(\gamma-\mu)y}\Big), \end{aligned}$$

where $\gamma := \sqrt{2\beta+\mu^2}$. The formulae (12) and (15) in Erdélyi [8, p. 246-247] for inverse Laplace transforms give us

$$\tilde{p}(t;x,y) = \frac{1}{2\mu\sqrt{2\pi t}}\, e^{\mu(y+x)-\frac{\mu^2 t}{2}}\Big(e^{-\frac{(x-y)^2}{2t}} + e^{-\frac{(x+y)^2}{2t}}\Big) + \Phi\Big(\frac{-y-x+\mu t}{\sqrt{t}}\Big). \tag{5}$$

This expression for $\tilde{p}$ is also (of course) obtained from (2) by differentiation. (We remark that in [6, p. 110] another, more complicated, expression is presented.)

2.2.2 V as a reflecting Brownian motion with negative drift

As stated in the introduction, our starting point is Reich's intuitively appealing formula. In this section we prove that Reich's definition leads directly to a characterization of the storage process as a stationary RBM($-\mu$) which is defined (notice that existence follows from Kolmogorov's extension theorem) in the following

Definition 2.7 Let m and $\tilde{p}$ be as in (4). The (continuous) process $X = \{X_t : t \in \mathbb{R}\}$ such that for any $t_1 < t_2 < \ldots < t_n$ and Borel sets $A_1, A_2, \ldots, A_n$

$$\begin{aligned}&\mathbf{P}(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n)\\&= \int_{A_1} m(dx_1) \ldots \int_{A_n} m(dx_n) \tilde{p}(t_2 - t_1; x_1, x_2) \ldots \tilde{p}(t_n - t_{n-1}; x_{n-1}, x_n)\end{aligned}$$

is called a stationary RBM($-\mu$).

Proposition 2.8 The stationary storage process $\{V_t : t \in \mathbb{R}\}$ is identical in law to a stationary RBM($-\mu$).

Proof By Proposition 2.2 it is enough to prove that $\{V_t : t \geq 0\}$ is identical in law to a RBM($-\mu$) started with the initial distribution

$$m(dx) = 2\mu e^{-2\mu x} dx.$$

Because m is the distribution of V_0 (cf. Proposition 2.4) it remains to prove that $\{V_t : t \geq 0\}$ given V_0 is a a RBM($-\mu$) started at V_0. To do this, recall from Theorem 2.6 that the process X defined for $t \geq 0$ via

$$X_t := V_0 + W_t - \mu t + Y_t, \quad Y_t := - \inf_{0 \leq s \leq t} \{(V_0 + W_s - \mu s) \wedge 0\},$$

where $\wedge$ ($\vee$) is the usual minimum (maximum) operator, is a RBM($-\mu$) started at V_0. Because

$$- \inf_{0 \leq s \leq t} \{(V_0 + W_s - \mu s) \wedge 0\} = \sup_{0 \leq s \leq t} \{(-V_0 - W_s + \mu s) \vee 0\},$$

we have for all $t \geq 0$

$$\begin{aligned} X_t &= V_0 + W_t - \mu t - \inf_{0\leq s\leq t}\{(V_0 + W_s - \mu s)\wedge 0\} \\ &= \sup_{0\leq s\leq t}\{(-V_0 - W_s + \mu s)\vee 0\} + V_0 + W_t - \mu t \\ &= V_0 \vee \sup_{0\leq s\leq t}\{-W_s + \mu s\} + W_t - \mu t \\ &= \sup_{-\infty<s\leq t}\{-W_s + \mu s\} + W_t - \mu t \\ &= V_t \end{aligned}$$

as claimed. □

Remark 2.9 The actual output from the storage during the time interval (s,t) is defined in a natural way as

$$U(s,t) := (W_t - W_s) - (V_t - V_s),$$

that is, the input minus the change in the contents of the storage. Using the definition of V we have

$$U_t := U(0,t) = \mu t + \inf_{s\leq t}\{W_s - \mu s\} - \inf_{s\leq 0}\{W_s - \mu s\},$$

from which the law of $\{U_t \,:\, t\geq 0\}$ can be deduced.

2.3 Symmetry and reversibility

It is interesting that although the definition of V is not symmetric in time the process V is anyway reversible. In this and subsequent sections $\mathbf{P}$ denotes the probability measure associated to the storage process V and $\mathbf{P}_x$ the probability measure associated to a RBM($-\mu$) X started at x.

Proposition 2.10 The process V run backward in time is identical in law to V, i.e.,

$$\{V_t \,:\, t\in\mathbb{R}\} \quad \sim \quad V^- := \{V_{-t} \,:\, t\in\mathbb{R}\}$$

(here and in the sequel the sign $\sim$ means that the object on the left hand side has the same distribution as the one on the right hand side).

Proof The claim is that for $t_1 < t_2 < \ldots < t_n$ and Borel sets $A_i,\ i = 1, 2, \ldots, n$

$$\begin{aligned} & \mathbf{P}(V_{t_1} \in A_1, V_{t_2} \in A_2, \ldots, V_{t_n} \in A_n) \\ = \; & \mathbf{P}(V_{-t_n} \in A_n, V_{-t_{n-1}} \in A_{n-1}, \ldots, V_{-t_1} \in A_1). \end{aligned} \tag{6}$$

From Proposition 2.8 and (4) we have

$$\mathbf{P}(V_t \in dy \,|\, V_s = x) = \mathbf{P}_x(X_{t-s} \in dy) = \tilde{p}(t-s; x, y)\, m(dy)$$

where $\tilde{p}(t; x, y) = \tilde{p}(t; y, x)$. Using this in (6) (take $n = 2$; the general case is similar) yields

$$\begin{aligned} \mathbf{P}(V_{t_1} \in A_1, V_{t_2} \in A_2) &= \int_{A_1} \mathbf{P}(V_{t_2} \in A_2 \,|\, V_{t_1} = x)\mathbf{P}(V_{t_1} \in dx) \\ &= \int_{A_1} m(dx) \int_{A_2} m(dy)\tilde{p}(t_2 - t_1; x, y) \\ &= \int_{A_2} m(dy) \int_{A_1} m(dx)\tilde{p}(t_2 - t_1; y, x) \\ &= \mathbf{P}(V_{-t_2} \in A_2, V_{-t_1} \in A_1). \end{aligned}$$

□

Corollary 2.11 Let $W^+ = \{W_t^+ : t \geq 0\},\ W_0^+ = 0$, be a BM($\mu$), $\mu > 0$. Then the process Z^+ defined by

$$Z_t^+ := W_t^+ - \inf_{s \geq t} W_s^+ \tag{7}$$

is (identical in law to) a stationary RBM($-\mu$).

Proof Because

$$\{W_t : t \in \mathbb{R}\} \quad \sim \quad \{W_{-t} : t \in \mathbb{R}\}$$

the reversed process V^- is identical in law to the process $V^\star$ defined by

$$V_t^\star := \sup_{s \geq t}\{W_t - W_s + \mu(t - s)\},$$

and so the claim follows from Proposition (2.8) and (2.10). □

Remark 2.12 The result in Corollary (2.11) can be seen as a companion to Skorokhod's result, Theorem 2.6, which says that if W^- is a BM$(-\mu)$ then the process Z^- defined by

$$Z_t^- := W_t^- - \inf_{s\le t} W_s^-,$$

is a RBM$(-\mu)$ started at 0. An alternative way to prove Corollary (2.11) is to use Williams' path decomposition theorem and Pitman's $2M - X$ -theorem. We remark also that that these two classical results have been applied in the context of queueing theory in the papers Harrison and Williamas [11] and O'Connell and Yor [14].

Symmetry in time leads to a symmetry in space in the sense of the following

Proposition 2.13 For all $s, t \in \mathbb{R},\ s \neq t$

$$\mathbf{P}(V_s > V_t) = 1/2.$$

Proof Using first time-reversibility and then stationarity we obtain

$$\mathbf{P}(V_s > V_t) = \mathbf{P}(V_{-s} > V_{-t}) = \mathbf{P}(V_t > V_s).$$

□

Remark 2.14 Because all diffusions have symmetric transition densities (w.r.t. their speed measures) the results in Proposition (2.10) and (2.13) hold for all stationary diffusions.

3 Busy periods (excursions)

In this section we present some additional results concerning the storage process V and find some distributions. The basic computation is straightforward: we take or compute the corresponding formula for RBM$(-\mu)$ started at x and integrate over x with respect to the stationary distribution. However, there are interesting phenomenona which make this analysis worthwhile. Recall that $\mathbf{P}$ stands for the probability measure associated with V and $\mathbf{P}_x$ for the probability measure associated with RBM$(-\mu)$ started at x.

3.1 Definition and the first characterization

Definition 3.1 For $t \in \mathbb{R}$ let

$$d_t := \inf\{s > t \,:\, V_s = 0\} \quad \text{and} \quad g_t := \sup\{s < t \,:\, V_t = 0\}.$$

The process $V^{(t)} := \{V_s \,:\, g_t \le s \le d_t\}$ is called the *on-going busy period* (or *excursion straddling* t).

Proposition 3.2 For a given t the process $V^{(t+)} := \{V_{t+s} \,:\, 0 \le s \le d_t - t\}$ is identical in law to a BM$(-\mu)$ which is started at time 0 with the initial distribution $m(dx) = 2\mu\, e^{-2\mu x}dx$ and killed at the first hitting time of 0, denoted H_0. Moreover, for $s > 0$

$$\mathbf{P}(V_{t+s} \in dy,\ t+s < d_t) = \mathbf{P}_y(H_0 > s)\, m(dy).$$

Proof The first statement follows directly from Proposition 2.8. For the second one let $\hat{p}$ be the symmetric transition density with respect to m (see Remark 2.14) of BM$(-\mu)$ killed at H_0. Then

$$\begin{aligned} \mathbf{P}(V_{t+s} \in dy,\ t+s < d_t) &= \int_0^\infty m(dx)\mathbf{P}_x(X_s \in dy, s < H_0) \\ &= \int_0^\infty m(dx)\hat{p}(s;x,y)m(dy) \\ &= m(dy)\int_0^\infty m(dx)\hat{p}(s;y,x) \\ &= \mathbf{P}_y(H_0 > s)\, m(dy). \end{aligned}$$

□

Remark 3.3 Let $\{\hat{P}_t \,:\, t \ge 0\}$ be the transition semigroup for BM$(-\mu)$ killed at H_0. Then $\{\rho_t\}$ with

$$\rho_t(dy) := \mathbf{P}_y(H_0 > t)\, m(dy)$$

is a (sub-)probability entrance law for $\{\hat{P}_t\}$, i.e., for all $s, t > 0$

$$\rho_{s+t} = \rho_s \hat{P}_t.$$

On the other hand, also $\nu_t(dy) := n_t(y)\, m(dy)$, where

$$n_t(y) := -\frac{\partial}{\partial t}\mathbf{P}_y(H_0 > t),$$

is an entrance law for $\{\hat{P}_t\}$. Consequently, we have the representation

$$\rho_t = \int_t^\infty \nu_s ds$$

which is related in a sense to the results of Getoor [9], section 8. Recall (e.g. from [9]) that the Itô excursion entrance law for RBM($-\mu$) for excursions from 0 is given by

$$\hat{\nu}_t(dy) := \nu_t(dy)/\mu.$$

3.2 The second characterization

Proposition 3.2 characterizes the law of the busy period straddling t via two conditionally independent (given V_t) and identically distributed processes; the first one running backward in time and the second one forward in time. In the next proposition another characterization is presented. This is also in terms of two processes but both are now running forward in time.

Proposition 3.4 Let B^+, $B_0^+ = 0$, be a three-dimensional Bessel process with drift $\mu > 0$, i.e., a diffusion associated to the differential operator

$$\mathcal{G} = \frac{1}{2}\frac{d^2}{dx^2} + \mu\frac{\cosh(\mu x)}{\sinh(\mu x)}\frac{d}{dx}, \quad x > 0.$$

Let η be an exponentially distributed random variable with parameter 2μ and define $\xi := -\eta$. Further, let $W^-, W_0^- = 0$, be a BM($-\mu$). For $x > 0$ introduce

$$\lambda_x := \sup\{s \,:\, B_s^+ = x\} \quad \text{and} \quad H_x := \inf\{s \,:\, W_s^- = -x\}.$$

Then

$$\{V_{g_t+s} : 0 \le s < d_t - g_t\} \quad \sim \quad \{Z_s^\circ \,:\, 0 \le s < \lambda_\eta + H_\xi\}$$

where

$$Z_t^\circ := \begin{cases} B_t^+ \,, & 0 \le t < \lambda_\eta \\ \eta + W_{t-\lambda_\eta}^- \,, & \lambda_\eta \le t < \lambda_\eta + H_\xi. \end{cases}$$

Proof Williams' time reversal result (see [19], and [16] p. 317) applied for W^- and B^+ gives

$$\{W^-_{H_0-s} \ : \ 0 \leq s < H_0\} \quad \sim \quad \{B^+_s \ : \ 0 \leq s < \lambda_x\},$$

where $W^-_0 = x$. Consequently, for $0 < s_1 < s_2 < \ldots < s_n$ we have

$$\begin{aligned} & \mathbf{P}(V_{g_t+s_1} \in dx_1, \ldots, V_{g_t+s_n} \in dx_n, \ g_t + s_n < t) \\ = & \int_0^\infty m(dx)\mathbf{P}_x(W^-_{H_0-s_n} \in dx_n, \ldots, W^-_{H_0-s_1} \in dx_1, \ s_n < H_0) \\ = & \int_0^\infty m(dx)\mathbf{P}^+_0(B^+_{s_1} \in dx_1, \ldots, B^+_{s_n} \in dx_n, \ s_n < \lambda_x), \end{aligned}$$

where $\mathbf{P}^+_0$ is the probability measure associated with B^+ started at 0. Interpreting here m as the distribution of η shows that the former part of Z° is as claimed. The complete description follows from Proposition 3.2 by conditional independence. □

It is interesting that it is possible to integrate above (with respect to x) and obtain an explicit characterization without the variable η. To state the result we introduce the diffusion playing the key role therein. Consider the ordinary differential operator

$$\mathcal{A}^\dagger u := \frac{1}{2}\frac{d^2}{dx^2}u + (\frac{1}{x} - \mu)\frac{d}{dx}u - \frac{\mu}{x}u.$$

The operator $\mathcal{A}^\dagger$ is an infinitesimal operator of a diffusion on $\mathbb{R}^+$ and it determines the diffusion uniquely because the boundary point 0 is entrance-not-exit (see [12]). We let $X^\dagger$ denote this diffusion. The life time ζ of $X^\dagger$ is a.s. finite.

Proposition 3.5 Let $X^\dagger$, $X^\dagger = 0$, be as above and W^- is a BM($-\mu$) started at 0 independent of $X^\dagger$, and define

$$Z^\dagger_t := \begin{cases} X^\dagger_t \ , & 0 \leq t < \zeta \\ X^\dagger_{\zeta-} + W^-_{t-\zeta} \ , & \zeta \leq t < \zeta + H_0, \end{cases}$$

where H_0 is the first hitting time of 0 for W^-. Then

$$\{V_{g_t+s} : 0 \leq s < d_t - g_t\} \quad \sim \quad \{Z^\dagger_s \ : \ 0 \leq s < \zeta + H_0\}.$$

Proof Let $x > 0$ and recall that the process

$$\{B_s^+ \ : \ 0 \le s < \lambda_x\},$$

is identical in law to the $h-$transform of B^+ with $h(y)$ being the probability that B^+ starting at y hits x, in other words,

$$h(y) = h_1(y) := \frac{G_0^+(y,x)}{G_0^+(0,x)},$$

where G_0^+ is the Green function (of index 0) of B^+. This means that for any $x_0 \ge 0$

$$\begin{aligned} & \mathbf{P}^+_{x_0}(B^+_{s_1} \in dx_1, \dots, B^+_{s_n} \in dx_n, \ s_n < \lambda_x) \\ = & \frac{1}{h_1(x_0)} \, \mathbf{P}^+_{x_0}(B^+_{s_1} \in dx_1, \dots, B^+_{s_n} \in dx_n) h_1(x_n) \\ = & \frac{G_0^+(x_n,x)}{G_0^+(x_0,x)} \, \mathbf{P}^+_{x_0}(B^+_{s_1} \in dx_1, \dots, B^+_{s_n} \in dx_n). \end{aligned}$$

It is, of course, possible to compute h_1 by finding the scale function of B^+. However, it is perhaps simpler to use the property of $h-$transforms which says, roughly speaking, that when making consecutive $h-$transforms it is only the last transform that is essential. To explain this let X denote a BM$(-\mu)$ killed at H_0. Then B^+ is an $h-$transform of X with $h(y) = h_2(y) := e^{2\mu y} - 1, \ y > 0$. Notice that h_2 is the scale function of X and B^+ can be viewed as X conditioned not to hit 0. The Green function of X (with respect to the speed measure) is

$$G_0(x,y) = G_0(y,x) = \frac{1}{2\mu} \, h_2(x), \quad x \le y,$$

and, hence, for $0 < x \le y$

$$G_0^+(x,y) = G_0^+(y,x) = \frac{G_0(x,y)}{h_2(x) h_2(y)} = \frac{1}{2\mu} \, \frac{1}{h_2(y)}.$$

Now we have (for $z > 0$)

$$\begin{aligned} h_3(z) \ &:= \ \int_0^\infty m(dx) \frac{G_0^+(z,x)}{G_0^+(0,x)} \\ &= \ \frac{2\mu z}{e^{2\mu z} - 1} = \frac{2\mu z}{h_2(z)}. \end{aligned}$$

The function h_3 is excessive for B^+ because it is obtained by integrating (minimal) excessive functions with respect to a finite measure (for general theory see Dynkin [7]; in [18] this theory is applied for linear diffusions). Moreover, because m is a probability measure we have $h_3(0) = 1$. Straightforward computation (or use of the formulae in [6, p. 33]) shows that the $h-$transform of B^+ with $h := h_3$ has the generator $\mathcal{A}^\dagger$, and the proof is complete. $\square$

Remark 3.6 The function $h_4(x) := x$ is excessive for BM$(-\mu)$ killed at H_0 (denoted X above) and the generator of the $h-$transform of X with $h = h_4$ is $\mathcal{A}^\dagger$. Notice that

$$h_4(x) = \mu\, \mathbf{E}_x(H_0).$$

In fact, the results above can be formulated very explicitly for all positively recurrent diffusions using scale functions, speed measures, and Green functions.

3.3 Some occupation times and lengths of busy periods

Introduce for a given t the occupation times

$$\alpha_t^+ := \int_{g_t}^{d_t} \mathbf{1}_{(V_t,\infty)}(V_s)\, ds$$

and

$$\alpha_t^- := \int_{g_t}^{d_t} \mathbf{1}_{(0,V_t)}(V_s)\, ds.$$

Proposition 3.7 The Laplace transform of the distribution of (α_t^-, α_t^+) is given by

$$\mathbf{E}\Big(\exp(-p\,\alpha_t^- - q\,\alpha_t^+)\Big) = \frac{2\mu}{\sqrt{2p+\mu^2} + \sqrt{2q+\mu^2}}. \tag{8}$$

In particular, α_t^- and α_t^+ are identical in law, and the common distribution is given by

$$\mathbf{P}(\alpha_t^- \in du) = \mathbf{P}(\alpha_t^+ \in du) = 2\mu(\frac{1}{\sqrt{2\pi u}} e^{-\mu^2 u/2} - \mu(1 - \Phi(\mu\sqrt{u})))\, du.$$

Proof Recall the formulae 2.6.1 in [6, p. 227]:

$$\mathbf{E}_x\Big(\exp\Big(-p\int_0^{H_0}\mathbf{1}_{(0,x)}(X_s)\,ds - q\int_0^{H_0}\mathbf{1}_{(x,\infty)}(X_s)\,ds\Big)\Big) =$$

$$\frac{\sqrt{2p+\mu^2}\;e^{\mu x}}{\sqrt{2p+\mu^2}\cosh(x\sqrt{2p+\mu^2})+\sqrt{2q+\mu^2}\sinh(x\sqrt{2p+\mu^2})}\,.$$

Because

$$\mathbf{E}\Big(\exp(-p\,\alpha_t^- - q\,\alpha_t^+)\Big) =$$

$$\int_0^\infty 2\mu\,e^{-2\mu x}\Big(\mathbf{E}_x\Big(\exp\Big(-p\int_0^{H_0}\mathbf{1}_{(0,x)}(X_s)\,ds - q\int_0^{H_0}\mathbf{1}_{(x,\infty)}(X_s)\,ds\Big)\Big)\Big)^2\,dx$$

the first claim follows by elementary integration, and from (8) it is clear that α_t^- and α_t^+ are identically distributed. To find the distribution use Erdélyi et al. [8, p. 233, (formula (4)]. □

Clearly,

$$d_t - g_t = \alpha_t^- + \alpha_t^+,$$

and we obtain from Proposition 3.7 the following

Corollary 3.8 The length of the busy period has the gamma distribution with parameters $\mu^2/2$ and $1/2$:

$$\mathbf{P}(d_t - g_t \in du) = \frac{\mu}{\sqrt{2\pi}}u^{-1/2}e^{-\mu^2 u/2}\,du.$$

Proposition 3.9 The remaining length of the busy period straddling t, that is, $\ell_t := d_t - t$ is identical in law with α_t^+ (and α_t^-).

Proof From Proposition 3.2 we have

$$\mathbf{P}(\ell_t \in du) = du\int_0^\infty n_u(y)\,m(dy).$$

Instead of integrating we compute the Laplace transform:

$$\begin{aligned}\mathbf{E}(e^{-\beta\ell_t}) &= \int_0^\infty 2\mu\, e^{-2\mu x}\mathbf{E}_x(e^{-\beta H_0})dx \\ &= \int_0^\infty 2\mu\, e^{-2\mu x}e^{-(\sqrt{2\beta+\mu^2}-\mu)x}dx \\ &= \frac{2\mu}{\sqrt{2\beta+\mu^2}+\mu}.\end{aligned}$$

which proves the claim. □

Remark 3.10 The common distribution of α_t^+, α_t^-, and ℓ_t, denoted F, can also be found in a number of other cases:

1) In the paper by Abate and Whitt [1] the main topic is to study the moments of RBM($-\mu$) the aim being to understand how RBM($-\mu$) approaches its steady state. In particular, they notice that

$$t \mapsto H(t) := 2\mu\,\mathbf{E}_0(X_t)$$

is a distribution function equal to F, and that H is also the distribution function of ℓ_t, the so called busy period distribution of the Brownian storage (see [1, Corollary 1.3.1 p. 566]). We refer to [1], [2], [3], [4], and [5] for further results in this direction and for references for earlier works. Especially, see Theorem 1 [5] for a local limit theorem in the framework of M/G/1 leading to the common distribution F (also called the RBM-equilibrium-time-to-emptiness by Abate and Whitt).

2) From [6, p. 204, Formula 1.5.3] it is seen that the occupation time

$$\int_0^\infty \mathbf{1}_{(-\infty,0)}(W_s+\mu s)\,ds,$$

where $W_0=0$ and $\mu>0$, has the distribution function F.

3) Let X° be a RBM(μ), $\mu>0$, starting at 0 and

$$\lambda_0 := \sup\{t\ :\ X_t^\circ = 0\}$$

the last exit time at 0. Then (cf. [6, p. 26] and references therein)

$$\mathbf{P}_0^\circ(\lambda_0\in ds) = \frac{p^\circ(t;0,0)}{G_0^\circ(0,0)}ds,$$

where $\mathbf{P}_0^\circ$ is the probability measure associated to X°, p° is the transition density with respect to the speed measure, and G_0° is the corresponding Green function (with index 0). Using formulae in [6, p. 110] it is seen that λ_0 has the distribution F.

4) Let L be the Brownian local time at 0 (defined as the limiting occupation time density with respect to the Lebesgue measure), and consider for $\mu > 0$ the process $T_t := L_t - \mu t$. Then it is proved in [13] that the random time

$$R := \inf\{t \ : \ T_t = \sup_{s\geq 0} T_s\}$$

has the distribution function F.

We have no probabilistic explanation for the fact that α_t^+, α_t^-, and ℓ_t are identical in law. However, Proposition 3.11 below explains why the means of these variables must be equal. Indeed,

$$\begin{aligned}
\mathbf{E}(\alpha_t^+) &= 2\,\mathbf{E}(\int_t^{d_t} \mathbf{1}_{(V_t,\infty)}(V_s)\,ds) \\
&= 2\int_0^\infty \mathbf{P}(V_{t+s} > V_t,\ t+s < d_t)\,ds \\
&= \mathbf{E}(\ell_t)
\end{aligned}$$

by the independence, shown in the next proposition, and Proposition 2.13.

Proposition 3.11 For a given t and $s > 0$ the events

$$\{V_{t+s} > V_t\} \quad \text{and} \quad \{d_t > t+s\}$$

are independent and, hence,

$$\mathbf{P}(V_{t+s} > V_t,\ s+t < d_t) = \frac{1}{2}\int_0^\infty m(dy)\,\mathbf{P}_y(H_0 > s).$$

Proof Consider (cf. Proposition 2.13)

$$\begin{aligned}
\mathbf{P}(V_{t+s} > V_t,\ t+s < d_t) &= \int_0^\infty m(dx)\mathbf{P}_x(X_s > x,\ s < H_0) \\
&= \int_0^\infty m(dx)\int_x^\infty m(dy)\hat{p}(s;x,y) \\
&= \int_0^\infty m(dy)\int_0^y m(dx)\hat{p}(s;y,x) \\
&= \mathbf{P}(V_{t+s} < V_t,\ t+s < d_t).
\end{aligned}$$

Consequently,

$$\mathbf{P}(d_t > t+s) = 2\,\mathbf{P}(V_{t+s} > V_t,\ t+s < d_t)$$

proving the claim. □

Remark 3.12 The occupation times

$$\tau_t^+ := \int_t^{d_t} \mathbf{1}_{(V_t,\infty)}(V_s)\,ds \quad \text{and} \quad \tau_t^- := \int_t^{d_t} \mathbf{1}_{(0,V_t)}(V_s)\,ds$$

can be analyzed similarly as α_t^+ and α_t^-. The distribution of τ_t^+ is given by

$$\mathbf{E}(e^{-\gamma\tau_t^+}) = \frac{2\mu}{\beta-\mu}\ln(1+\frac{\beta-\mu}{2\mu}),$$

but it does not seem to be possible to give a closed form expression for the Laplace transform of τ_t^-. Anyway, it can be proved that τ_t^+ and τ_t^- are not identically distributed. On the other hand, we have obviously

$$\mathbf{E}(\tau_t^+) = \mathbf{E}(\tau_t^-) = \frac{1}{2}\mathbf{E}(\ell_t) = \frac{1}{4\mu^2}.$$

3.4 Maxima within busy periods

Finally, we study the maximum of the storage process during a busy period. Define (for a given $t \in \mathbf{R}$)

$$M_t := \sup\{V_s\ :\ g_t \le s \le d_t\}$$

and

$$m_t := \sup\{V_s\ :\ t \le s \le d_t\}.$$

Proposition 3.13 The distributions of M_t and m_t are given by

$$\mathbf{P}(M_t > y) = \frac{2\,e^{-2\mu y}}{(1-e^{-2\mu y})^2}(2\mu y - (1-e^{-2\mu y})), \quad y > 0,$$

and

$$\mathbf{P}(m_t > y) = \frac{2\mu y\, e^{-2\mu y}}{1-e^{-2\mu y}}, \quad y > 0.$$

Further,

$$\mathbf{E}(M_t) = \frac{1}{\mu} > \mathbf{E}(m_t) = \frac{\pi^2}{2\mu} > \mathbf{E}(V_t) = \frac{1}{2\mu}.$$

Proof For m_t we have

$$\mathbf{P}(m_t < y) = \int_0^y 2\mu\, e^{-2\mu x}\mathbf{P}_x(H_0 < H_y)\, dx,$$

and, because

$$\mathbf{P}_x(H_0 < H_y) = \frac{e^{2\mu y} - e^{2\mu x}}{e^{2\mu y} - 1}$$

the claimed distribution is obtained by integration. To compute the mean use

$$(1 - e^{-2\mu y})^{-1} = \sum_{k=0}^{\infty} e^{-2\mu k y}$$

in

$$\mathbf{E}(m_t) = \int_0^{\infty} \mathbf{P}(m_t > y) dy.$$

For M_t we have, because $\{V_{t-s} \,:\, 0 \le s \le t - g_t\}$ and $\{V_{t+s} \,:\, 0 \le s \le d_t - t\}$ are identical in law and conditionally independent given V_t,

$$\mathbf{P}(M_t < y) = \int_0^y 2\mu\, e^{-2\mu x}(\mathbf{P}_x(H_0 < H_y))^2\, dx$$

which leads to the desired distribution. The mean can be computed by integration by parts. □

Remark 3.14 The Laplace transform of m_t has also a clean expression. To compute it notice that

$$\begin{aligned}\int_0^{\infty} e^{-\alpha y}\mathbf{P}(m_t > y)dy &= \int_0^{\infty} 2\mu\, y\, e^{-(2\mu+\alpha)y} \sum_{k=0}^{\infty} e^{-2\mu k y}\, dy \\ &= \frac{1}{2\mu}\sum_{k=1}^{\infty}(k + \frac{\alpha}{2\mu})^{-2}.\end{aligned}$$

Using the notation (cf. Erdélyi et.al. [8, p. 370]; notice our summation starts from 1) associated to Riemann's zeta function

$$\zeta(z, a) := \sum_{n=1}^{\infty}(n + a)^{-z}$$

we have

$$\int_0^\infty e^{-\alpha y}\mathbf{P}(m_t > y)dy = \frac{1}{2\mu}\zeta(2, \frac{\alpha}{2\mu})$$

For $\alpha = 0$ we obtain the mean; recall that $\zeta(2, 0) = \pi^2/6$ is the value of the Riemann's zeta function at 2. The Laplace transform of m_t takes now the form

$$\mathbf{E}(e^{-\alpha m_t}) = 1 - \frac{\alpha}{2\mu}\zeta(2, \frac{\alpha}{2\mu}).$$

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