PRACTICAL NORMAL FORM COMPUTATIONS

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CADE 2007
Turku
OBJECTS:  Local ODES

\[ \dot{x} = f(x), \quad f(0) = 0, \]
sufficiently differentiable;  
Taylor:

\[ f(x) = Bx + f_2(x) + f_3(x) + \cdots \]

\( (B \text{ linear, each } f_j \text{ homogeneous of degree } j) \)

OBJECTIVE:  SIMPLIFY!

Take (analytic) “near-identity” map

\[ H(x) = x + h_2(x) + \cdots. \]

Then there is unique vector field

\[ f^* = B + f^*_2 + \cdots \]

such that identity

\[ (\dagger) \quad DH(x)f^*(x) = f(H(x)) \]

holds. (\( f^* \) and \( f \) related by \( H \).)

POSSIBLE STRATEGY:  Degree by degree:

“Normalize” \( f_2, f_3, \ldots \) successively.
HOMOLOGICAL EQUATION

SITUATION:

\[ f = B + f_2 + \cdots + f_{r-1} + f_r + \cdots, \]
with \( f_2, \ldots, f_{r-1} \) “satisfactory”.

Normalize degree \( r \) term:

\[ H(x) = x + h_r(x) + \cdots. \]

Then \( f^* = B + f_2 + \cdots + f_{r-1} + f_r^* + \cdots, \) with

\[ [B, h_r] = f_r - f^*_r \quad \text{ (homological equation)} \]

(Lie bracket: \([p, q](x) := Dq(x)p(x) - Dp(x)q(x)\).)

Equation on the finite dimensional vector space \( \mathcal{P}_r \) of homogeneous vector fields of degree \( r \).

(Note: \( \text{ad } B = [B, \cdot] \) sends \( \mathcal{P}_r \) to itself.)

GENERAL OBSERVATION:

If \( W \) is any subspace of \( \mathcal{P}_r \) so that \( \mathcal{P}_r = \text{im } (\text{ad } B) + W \) then one can choose \( f_r^* \in W \).

HIGHER ORDER TERMS:

e.g. \( H = \exp(h_r) \) (time-one-map); then

\[ f^* = \exp(\text{ad } h_r)(f) \]
\[ [B, h_r] = f_r - f_r^* \quad \text{on} \quad P_r \]

Decomposition \( B = B_s + B_n \)  
(semisimple + nilpotent). Then  
\[ \text{ad } B = \text{ad } B_s + \text{ad } B_n \]

\textbf{POINCARÉ-DULAC:}  
Choose \( W = \ker (\text{ad } B_s) \), so \( [B_s, f_r^*] = 0 \).

\textbf{EXAMPLE:}  
\( B = B_s = \text{diag} (\lambda_1, \ldots, \lambda_n) \),  
\( p(x) = x_1^{m_1} \cdots x_n^{m_n} e_j \). Then  
\[ [B, p] = \left( m_1 \lambda_1 + \cdots + m_n \lambda_n - \lambda_j \right) \cdot p \]

Homological equation easy to solve in this case.  
(Moreover: Role of eigenvalues becomes transparent.)

\textbf{PROBLEM:}  
What if eigenvalues of \( B \) are not known explicitly?
\[ [B, h_r] = f_r - f_r^* \]

For sake of simplicity: \( B = B_s \)

**Required:** Polynomial
\[
p(\tau) = \tau^m + \alpha_1 \tau^{m-1} + \cdots + \alpha_{m-1} \tau + \alpha_m
\]
that annihilates \( \text{ad} \ B \) on \( \mathcal{P}_r \).

W.l.o.g.: \( \alpha_{m-1} \neq 0 \) if \( \alpha_m = 0 \) (semisimplicity)

**PROPOSITION:**

(a) In case \( \alpha_m \neq 0 \), \( f_r^* = 0 \) and
\[
h_r = -\frac{1}{\alpha_m} \left( (\text{ad} \ B_s)^{m-1} + \cdots + \alpha_{m-1} \text{id} \right) (f_r)
\]

(b) In case \( \alpha_m = 0 \):
\[
h_r = -\frac{1}{\alpha_{m-1}} \left( (\text{ad} \ B_s)^{m-2} + \alpha_1 (\text{ad} \ B_s)^{m-3} + \cdots + \alpha_{m-2} \text{id} \right) (f_r),
\]
\[
f_r^* = f_r - [B, h_r]
\]

(Idea: Use projections to kernel and image of \( \text{ad} \ B \); these can be obtained from polynomial \( p \).)

**Note** on practical matters: Use Horner type evaluation.
FINDING ANNIHILATING POLYNOMIALS

\[ B = B_s \quad \text{(for sake of simplicity)}, \text{ eigenvalues} \lambda_1, \ldots, \lambda_m \text{ (distinct); minimum polynomial} \]

\[(\tau - \lambda_1) \cdots (\tau - \lambda_m) = \tau^m - \sigma_1 \tau^{m-1} + \cdots + (-1)^m \sigma_m,\]

with \( \sigma_1 = \lambda_1 + \cdots + \lambda_m, \ldots, \sigma_m = \lambda_1 \cdots \lambda_m \)
elementary symmetric functions in the \( \lambda_i \).

OBSERVATION: If \( \beta_1, \ldots, \beta_s \) are the eigenvalues of \( \text{ad} \ B \) on \( P_r \) then the \( \beta_i + \lambda_j \ (1 \leq i \leq s, 1 \leq j \leq m) \) are the eigenvalues of \( \text{ad} \ B \) on \( P_{r+1} \).

PROPOSITION: Let \( q(\tau) \) annihilate \( \text{ad} \ B \) on \( P_r \). Then

\[ \hat{q}(\tau) = \prod_{1 \leq j \leq m} q(\tau - \lambda_j) \]

annihilates \( \text{ad} \ B \) on \( P_{r+1} \). Moreover, \( \hat{q} \) is symmetric in \( \lambda_1, \ldots, \lambda_m \) and can therefore be expressed as polynomial in \( \tau \) and \( \sigma_1, \ldots, \sigma_m \).

(Routine task for algorithmic algebra!)
WHAT HAS BEEN GAINED?

- Normal forms are not interesting by themselves!
- Why even insist on Poincaré-Dulac?

Answer to second question:

Poincaré-Dulac has built-in symmetries; allows canonical reduction via invariants.

FINDING INVARIANTS: \((B = B_s)\)

Lie derivative \(L_B\) acts on scalar-valued polynomials, and also on \(S_r\) (space of homogeneous polynomials of degree \(r\)). Invariants of \(B\) are just the elements of the kernel of \(L_B\).

Variant of above procedure: Find annihilating polynomial for \(L_B\) on \(S_r\), and projection to kernel of \(L_B\).

REDUCTION: \(f\) in Poincaré-Dulac form (truncated); \(\varphi_1, \ldots, \varphi_s\) generators of invariant algebra of \(B\).

Then each \(L_f(\varphi_i)\) is invariant of \(B\); thus \(L_f(\varphi_i) = \gamma_i(\varphi_1, \ldots, \varphi_s)\) for some polynomial \(\gamma_i\).

This yields reduced vector field \(g = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_s \end{pmatrix}\).
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*On normal form computations.*

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WEB PAGE:

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