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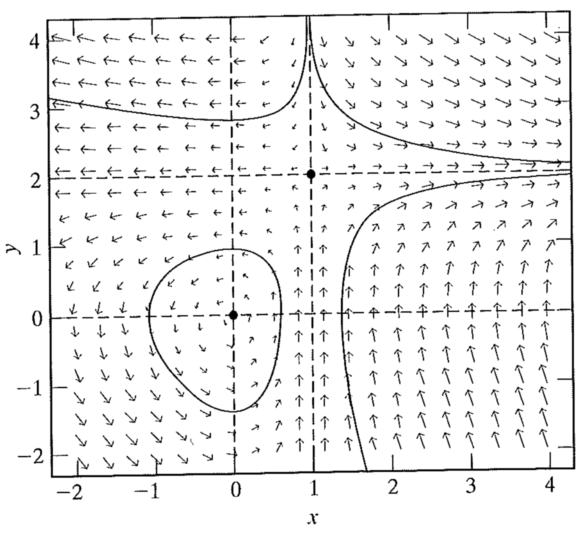

5.7 Periodic Solutions

Analytical results concerning periodic solutions for two-dimensional autonomous systems $dx/dt = f(x, y)$ and $dy/dt = g(x, y)$ are stated in this section. The first result is known as the Poincaré-Bendixson Theorem. This theorem states that bounded solutions whose limiting set does not contain any equilibria must approach a periodic solution. There are two other important theorems known as Bendixson's and Dulac's criteria. Each of these theorems give a criterion such that if it is satisfied, then the system will not have any periodic solutions.

5.7.1 Poincaré-Bendixson Theorem

Some terminology and notation are introduced in regard to a phase plane analysis. The term "trajectory" is used synonymously with *orbit* in the phase plane. The notation $\Gamma(X_0, t)$ is used to denote a solution trajectory as a function of time t beginning at the initial point $X_0 = (x(t_0), y(t_0)) = (x_0, y_0)$. In addition, $\Gamma^+(X_0, t)$ denotes that part of the solution trajectory where $t \geq t_0$, a *positive orbit*, and $\Gamma^-(X_0, t)$ denotes that part of the solution where $t \leq t_0$, a *negative orbit*. If solutions are bounded, then their negative and positive orbits approach limiting sets as $t \to -\infty$ or as $t \to +\infty$. The *α-limit set*, denoted $\alpha(X_0)$, refers to the set of points in the plane that are approached by the negative orbit, $\Gamma^-(X_0, t)$, as $t \to -\infty$ [i.e., $(x_l, y_l) \in \alpha(X_0)$ iff there exists a sequence of decreasing times $\{t_i\}_{i=1}^{\infty}, t_i \to -\infty$ as $i \to \infty$, such that $\lim_{i\to\infty}(x(t_i), y(t_i)) = (x_l, y_l)$]. The *ω-limit set*, denoted $\omega(X_0)$, refers to the set of points in the plane that are approached by the positive orbit, $\Gamma^+(X_0, t)$, as $t \to \infty$.

A very important result in the theory of two-dimensional autonomous systems is known as the Poincaré-Bendixson Theorem. This theorem states conditions for existence of periodic solutions to the system (5.5). The names Poincaré and Bendixson refer to the contributions made by the well-known French mathematician Jules Henri Poincaré (1854–1912) and the Swedish mathematician Ivar O. Bendixson (1861–1935).

Theorem 5.6 **(Poincaré-Bendixson Theorem).** *Let $\Gamma^+(X_0, t)$ be a positive orbit of (5.5) that remains in a closed and bounded region of the plane. Suppose the ω-limit set does not contain any equilibria. Then either*

(i) $\Gamma^+(X_0, t)$ is a periodic orbit $(\Gamma^+(X_0, t) = \omega(X_0))$ or
(ii) the ω-limit set, $\omega(X_0)$, is a periodic orbit. □

For a proof of this result, consult Coddington and Levinson (1955). An important consequence of this theorem is known as the Poincaré-Bendixson trichotomy, which states that bounded solutions containing only a finite number of equilibria can behave in one of only three ways (Coddington and Levinson, 1955; Smith and Waltman, 1995).

Theorem 5.7 **(Poincaré-Bendixson Trichotomy).** *Let $\Gamma^+(X_0, t)$ be a positive orbit of (5.5) that remains in a closed and bounded region B of the plane. Suppose B contains only a finite number of equilibria. Then the ω-limit set takes one of the following three forms:*

(i) $\omega(X_0)$ is an equilibrium.
(ii) $\omega(X_0)$ is a periodic orbit.
(iii) $\omega(X_0)$ contains a finite number of equilibria and a set of trajectories Γ_i whose α- and ω-limit sets consist of one of these equilibria for each trajectory Γ_i. □

An important assumption in both of these theorems is that solutions are bounded. In Case (ii), if $\Gamma^+(X_0, t) \neq \omega(X_0)$ but approaches the periodic orbit, then the periodic orbit may be a *limit cycle*. In Case (iii), the limiting set is referred to as a *cycle graph*. The cycle graph may consist of either an equilibrium and a *homoclinic orbit* (connecting an equilibrium to itself) or several equilibria and *heteroclinic orbits* (connecting two different equilibria). Examples of ω-limit sets are graphed in Figure 5.10.

An important fact is that a periodic orbit must enclose at least one equilibrium point. A periodic orbit changes direction as it follows a closed curve in the *x*-*y* plane and a change in direction can only occur at an equilibrium point. Another important fact concerning periodic orbits is that if there exists exactly one equilibrium point inside a periodic orbit, it cannot be a saddle point. But it can be a node or a spiral. The direction of flow around a saddle point does not allow periodic orbits to encircle it. The direction of flow around any closed curve in the plane can be classified according to the *index* of a closed curve. [See, for example, Coddington and Levinson (1955) or Strogatz (2000).]

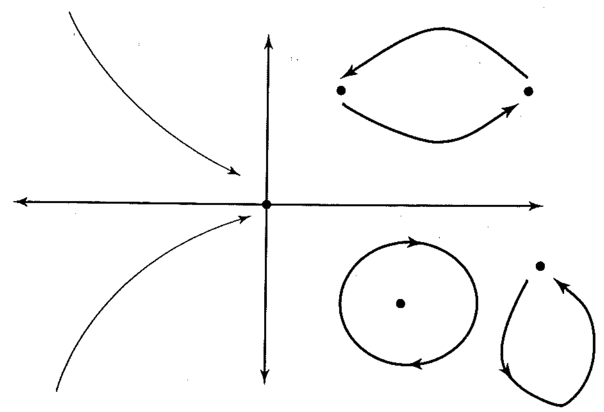

Figure 5.10 Examples of ω-limit sets in the phase plane.

xample 5.14 Consider the following nonlinear system:

$$\frac{dx}{dt} = 8x - y^2, \quad \frac{dy}{dt} = -y + x^2. \tag{5.6}$$

The x-nullcline is $y^2 = 8x$ and the y-nullcline is $y = x^2$. The nullclines intersect at two equilibria $(0, 0)$ and $(2, 4)$. On the x-nullcline, dy/dt satisfies

$$\left.\frac{dy}{dt}\right|_{y^2=8x} = -y + \frac{y^4}{64} = y\left(-1 + \frac{y^3}{64}\right).$$

When $y < 0$ or $y > 4$, then $dy/dt > 0$ and when $0 < y < 4$, $dy/dt < 0$. On the y-nullcline, dx/dt satisfies

$$\left.\frac{dx}{dt}\right|_{y=x^2} = 8x - x^4 = x(8 - x^3).$$

When $x < 0$ or $x > 2$, then $dx/dt < 0$, and when $0 < x < 2$, then $dx/dt > 0$. The direction of flow along the nullclines is sketched in Figure 5.11.

Next, we determine the behavior near the equilibria. The Jacobian matrix

$$J = \begin{pmatrix} 8 & -2y \\ 2x & -1 \end{pmatrix}.$$

At the origin,

$$J(0, 0) = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}.$$

Because one eigenvalue is positive ($\lambda_1 = 8$) and one is negative ($\lambda_2 = -1$), the origin is a saddle point. Solutions move away from the origin along the x-axis (*unstable manifold*) and move toward the origin along the y-axis (*stable manifold*). This behavior can be seen if we solve the linear system $dZ/dt = J(0, 0)Z$,

$$Z = c_1 e^{8t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

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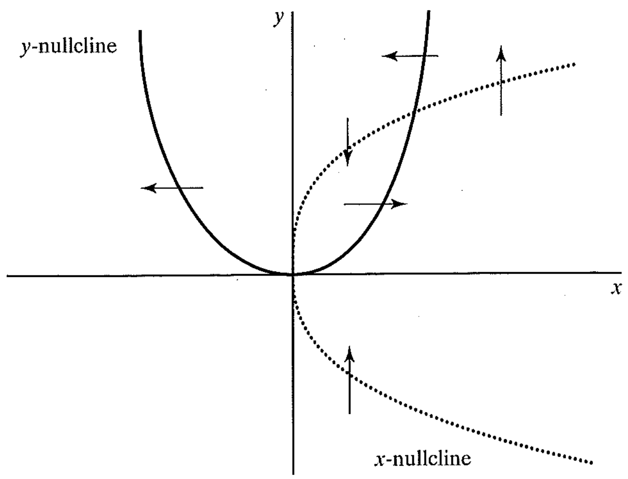

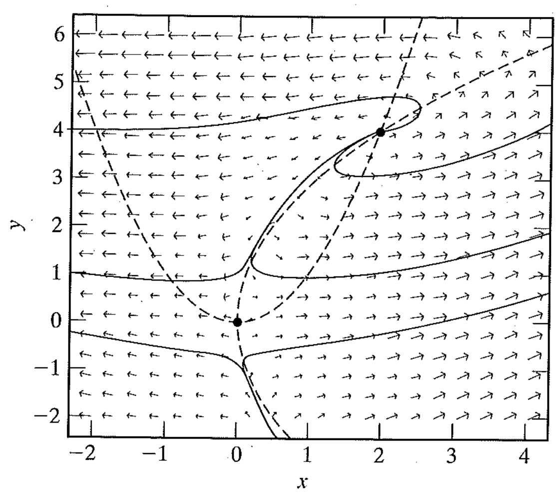

Figure 5.12 Direction field and some solution trajectories for the system (5.6). The nullclines are the dashed curves and the solutions are the solid curves.

At the equilibrium $(2, 4)$,

$$J(2, 4) = \begin{pmatrix} 8 & -8 \\ 4 & -1 \end{pmatrix}.$$

The characteristic equation is $\lambda^2 - 7\lambda + 24 = 0$ or $\lambda_{1,2} = 7/2 \pm i\sqrt{47}/2$. The equilibrium $(2, 4)$ is an unstable spiral point. The direction field and some solution trajectories are graphed in Figure 5.12 for the system (5.6). ■

The Poincaré-Bendixson Theorem can be applied to the nonlinear system in the last example if solutions are bounded. But it is easy to show that this is not the case for system (5.6). For $x(0) < 0$, $dx/dt < 0$ so that $x(t)$ is decreasing. In addition, $dx/dt < 8x$. Thus, $x(t) < x(0)e^{8t} \to -\infty$ as $t \to \infty$. Because solutions are not bounded, the Poincaré-Bendixson Theorem cannot be applied to check for periodic solutions. However, the next two results can be applied.

5.7.2 Bendixson's and Dulac's Criteria

Two important mathematical results give sufficient conditions that rule out the possibility of periodic solutions. They are Bendixson's criterion and Dulac's criterion. First, we define a simply connected set. A *simply connected set* $D \subset \mathbf{R}^2$ is a connected set having the property that every simple closed curve in D can be continuously shrunk (within D) to a point (Rudin, 1974). For example, the entire plane, $\mathbf{R}^2$, is a simply connected set. Geometrically, a simply connected set is one without any holes.

Theorem 5.8 **(Bendixson's Criterion).** *Suppose D is a simply connected open subset of $\mathbf{R}^2$. If the expression $div(f, g) \equiv \partial f/\partial x + \partial g/\partial y$ is not identically zero and does not change sign in D, then there are no periodic orbits of the autonomous system (5.5) in D.*

Proof Assume that there is a periodic solution C (a simple closed curve) in the simply connected region D. Let S denote the interior of C. When C is transversed counterclockwise, Green's Theorem in the plane (integration by parts in two dimensions) gives the following identity:

$$\int_C f(x,y)\,dy - g(x,y)\,dx = \iint_S \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dx\,dy. \tag{5.7}$$

Note that the right-hand side does not equal zero by hypothesis. The autonomous system satisfies

$$\frac{dx}{dy} = \frac{f(x,y)}{g(x,y)} \quad \text{or} \quad g(x,y)\,dx = f(x,y)\,dy.$$

Thus, the integral on the left side of (5.7) must be zero, which leads to a contradiction. □

nple 5.15 Bendixson's criterion can be applied to the system in Example 5.14. In this example, $f(x,y) = 8x - y^2$ and $g(x,y) = -y + x^2$. Let D be any open region in $\mathbf{R}^2$. Then $\text{div}(f,g) = 8 - 1 = 7 \neq 0$. Bendixon's criterion implies there are no periodic solutions in D. ■

A simple but important generalization of Bendixson's criterion is known as Dulac's criterion.

orem 5.9 **(Dulac's Criterion).** *Suppose D is a simply connected open subset of $\mathbf{R}^2$ and $B(x,y)$ is a real-valued C^1 function in D. If the expression*

$$\text{div}(Bf, Bg) = \frac{\partial(Bf)}{\partial x} + \frac{\partial(Bg)}{\partial y}$$

is not identically zero and does not change sign in D, then there are no periodic solutions of the autonomous system (5.5) in D. □

The function B is called a *Dulac function*. Dulac's criterion simplifies to Bendixson's criterion in the special case $B(x,y) \equiv 1$. There is no general method for determining an appropriate Dulac function for a given system. The difficulty in finding a Dulac function is similar to the difficulty in finding an appropriate "integrating factor" when solving differential equations (Hale and Koçak, 1991). Note that Dulac's and Bendixson's criteria give sufficient but not necessary conditions for the nonexistence of periodic solutions. If neither of these criteria are satisfied, there may or may not be periodic solutions.

nple 5.16 Suppose f and g are linear functions:

$$\frac{dx}{dt} = ax + by,$$

$$\frac{dy}{dt} = cx + dy.$$

Applying Bendixson's criterion, $\partial f/\partial x + \partial g/\partial y = a + d$. If $a + d \neq 0$, then there are no periodic solutions in the entire plane. But if $a + d = 0$, then Bendixson's criterion does not apply. The following linear system

$$\frac{dx}{dt} = y,$$

$$\frac{dy}{dt} = -x,$$

has $a = 1$ and $d = -1$, so that Bendixson's criterion does not apply. By separating variables and solving for x and y, this system satisfies $x^2 + y^2 = C = \text{constant}$; the origin is a center. Every solution, not beginning at the origin, is a periodic solution. The following linear system

$$\frac{dx}{dt} = y,$$

$$\frac{dy}{dt} = x,$$

has $a = 0 = d$, so again Bendixson's criterion does not apply. However, for this system, there do not exist any periodic solutions; the origin is a saddle point ($x^2 - y^2 = C = \text{constant}$). ■

Example 5.17 Consider the predator-prey model, where prey and predator grow logistically in the absence of the other species,

$$\frac{dx}{dt} = x(1 - ax - by),$$

$$\frac{dy}{dt} = y(1 + cx - dy),$$

where $a, b, c, d > 0$. Let $B(x,y) = 1/(xy)$. Note that B is continuously differentiable in the positive quadrant, $D = \{(x,y)|x > 0, y > 0\}$. Thus, $\text{div}(Bx(1 - ax - by), By(1 + cx - dy)) = -a/y - d/x < 0$ in D. Dulac's criterion implies there does not exist any periodic solutions in D. ■

The Poincarè-Bendixson Theorem and Dulac's and Bendixon's criteria apply only in two dimensions. However, there is a generalization of Dulac's criteria to three dimensions in some special cases (Busenberg and van den Driessche, 1990). This generalization involves finding a vector function g such that along solutions, the dot product of the curl of g and the unit normal vector, on the surface of a region in $\mathbf{R}^3_+$, is negative.

5.8 Bifurcations

If a parameter is allowed to vary, the dynamics of the differential system may change. An equilibrium can become unstable and a periodic solution may appear or a new stable equilibrium may appear making the previous equilibrium unstable. The value of the parameter at which these changes occur is known as a *bifurcation value* and the parameter that is varied is known as the *bifurcation parameter*. We discuss several types of bifurcations: saddle node, transcritical, pitchfork, and Hopf bifurcations. The first three types of bifurcations occur in scalar and in systems of differential equations. The fourth type, Hopf, does not occur in scalar differential equations because this type of bifurcation involves a change to a periodic solution. Scalar autonomous differential equations cannot have periodic solutions. Excellent introductions to the theory of nonlinear dynamical systems and bifurcation theory in differential equations include the books by Hale and Koçak (1991) and Strogatz (2000).

5.8.1 First-Order Equations

First, we discuss bifurcations in the case of scalar differential equations. Consider the scalar differential equation

$$\frac{dx}{dt} = f(x, r), \tag{5.8}$$

where r is the bifurcation parameter and $\bar{x}(r)$ is an equilibrium solution which depends on r. There are three different types of bifurcations:

I. saddle node
II. pitchfork
III. transcritical

These three types of bifurcations occur in scalar difference equations also. However, in scalar difference equations, there is additional type of bifurcation known as a period-doubling bifurcation.

At the bifurcation value $\bar{r}$, it is the case that the equilibrium changes stability. In particular, for $r = \bar{r}$ and $x = \bar{x}(\bar{r})$,

$$\left.\frac{df(x,r)}{dx}\right|_{(x,r)=(\bar{x}(\bar{r}),\bar{r})} = 0. \tag{5.9}$$

We discuss briefly the dynamics for each of three types of bifurcations for scalar differential equations. The bifurcation dynamics are similar to the dynamics in the case of difference equations, discussed in Chapter 2. In a saddle node bifurcation, as the bifurcation parameter passes through the bifurcation point, two equilibria disappear, so that there are no equilibria afterward. One of the two equilibria is stable and the other one is unstable, before they disappear. This type of bifurcation is sometimes referred to as a *blue sky bifurcation* (Strogatz, 2000) because equilibria appear as "out of the clear blue sky." In a pitchfork bifurcation, there are two stable equilibria separated by an unstable equilibrium. A system where there are two different stable equilibria is said to have the property of *bistability*. When the bifurcation point is passed, there is only one stable equilibrium. This type of bifurcation is referred to as a *supercritical pitchfork bifurcation*. There is also a *subcritical pitchfork bifurcation*. In a subcritical pitchfork bifurcation, the stability is the reverse of the supercritical bifurcation, that is, there are two unstable equilibria separated by a stable equilibrium, until the bifurcation point is passed. Then there is only one unstable equilibrium. The diagram looks like a "pitchfork." The diagram in Figure 2.10 II is a supercritical pitchfork bifurcation. In a transcritical bifurcation, there are two equilibria, one stable and one unstable. When the bifurcation point is passed, there is an exchange of stability; the unstable equilibrium becomes stable and the stable one becomes unstable.

The following three examples are canonical examples of these three types of bifurcations.

I. $\dfrac{dx}{dt} = r + x^2$

II. $\dfrac{dx}{dt} = rx - x^3$

III. $\dfrac{dx}{dt} = rx + x^2$

In each case, the bifurcation value is at $r = 0$. At $r = 0$, there is a change in the stability of the equilibrium. The criterion in (5.9) is satisfied for $\bar{r} = 0$ and $\bar{x}(\bar{r}) = 0$. A bifurcation diagram illustrating bifurcations of type I, II, and III is the same as the one for difference equations. See Chapter 2, Figure 2.10.

Example 5.18 Consider the canonical differential equation of type I: $dx/dt = r + x^2 = f(x, r)$. The equilibria satisfy $x^2 = -r$ or $\bar{x}(r) = \pm\sqrt{-r}$. When $r > 0$, there are no equilibria. When $r = 0$, there is one equilibrium. Finally, when $r < 0$, there are two equilibria. Also, $df(x, r)/dx = 2x$ evaluated at $\bar{x}$(r) equals $\pm 2\sqrt{-r}$. The positive equilibrium is unstable and the negative one is stable. There is a saddle node bifurcation at $\bar{r} = 0$. ■

These types of bifurcations also occur in higher-dimensional systems of differential equations. For example, in the two-dimensional system with equations

$$\frac{dx}{dt} = r + x^2,$$

$$\frac{dy}{dt} = -y,$$

there is a saddle node bifurcation at $r = 0$.

5.8.2 Hopf Bifurcation Theorem

A fourth type of bifurcation occurs in systems of differential equations consisting of two or more equations. This fourth type is known as a *Hopf bifurcation*. It is also referred to as a *Poincaré-Andronov-Hopf bifurcation* (Hale and Koçak, 1991) to acknowledge the contributions to the theory by French mathematician Jules Henri Poincaré (1854–1912), Russian mathematician Alexander A. Andronov (1901–1952), and German mathematician Heinz Hopf (1894–1971). We have seen in Chapter 3 that a similar type of bifurcation occurred in a predator-prey system modeled by a system of difference equations (known as a Neimark-Sacker bifurcation).

The Hopf Bifurcation Theorem stated here is for a system of two differential equations. There is a Hopf Bifurcation Theorem for higher dimensions also (see Marsden and McCracken, 1976). This Hopf Bifurcation Theorem states sufficient conditions for the existence of periodic solutions. As one parameter is varied, the dynamics of the system change from a stable spiral to a center to an unstable spiral. The eigenvalues of the linearized system change from having negative real part to zero real part to positive real part. Under certain conditions, there exist periodic solutions.

Consider a system of autonomous differential equations given by

$$\frac{dx}{dt} = f(x, y, r) \quad \text{and} \quad \frac{dy}{dt} = g(x, y, r), \tag{5.10}$$

where the functions f and g depend on the bifurcation parameter r. Suppose there exists an equilibrium $(\bar{x}(r), \bar{y}(r))$ of system (5.10) and the Jacobian matrix evaluated at this equilibrium has eigenvalues $\alpha(r) \pm i\beta(r)$. In addition, suppose a change in stability occurs at the value of $r = r^*$, where $\alpha(r^*) = 0$. If $\alpha(r) < 0$ for values of r close to r^* but for $r < r^*$ and if $\alpha(r) > 0$ for values of r close to r^* but for $r > r^*$ (also $\beta(r^*) \neq 0$), then the equilibrium changes from a stable spiral to an unstable spiral as r passes through r^*. The Hopf Bifurcation Theorem states that there exists a periodic orbit near $r = r^*$ for any neighborhood of the equilibrium in $\mathbf{R}^2$. The parameter r is the *bifurcation parameter* and r^* is the *bifurcation value*. The theorem is valid only when the bifurcation parameter has values close to the bifurcation value.

Before we state the theorem, a simple example is presented which exhibits a Hopf bifurcation.

Example 5.19 Consider the linear system

$$\frac{dx}{dt} = rx - y,$$

$$\frac{dy}{dt} = x + ry.$$

The origin is an equilibrium. The trace and determinant of the Jacobian matrix evaluated at the origin are $2r$ and $r^2 + 1$, respectively. Since the discriminant of the Jacobian matrix is negative, $(2r)^2 - 4r^2 - 4 = -4$, the eigenvalues are $r \pm i$. If $r < 0$, the origin is a stable spiral. If $r = 0$, the origin is a center, and if $r > 0$, it is an unstable spiral. The bifurcation value is at $r = r^* = 0$. Recall the stability diagram, where stability is graphed as a function of the trace τ and determinant δ. The bifurcation in this example occurred because r crossed the δ-axis where $\delta > 0$. A Hopf bifurcation occurs. As the bifurcation parameter r increases through the bifurcation value $r^* = 0$, the equilibrium $(0, 0)$ changes from a stable spiral to a neutral center to an unstable spiral. There are infinitely many periodic solutions at the bifurcation value $r^* = 0$. Solutions to $dx/dt = -y$ and $dy/dt = x$ are of the form $x^2(t) + y^2(t) = c$, where c is a constant that depends on initial conditions. ■

The linear example illustrates the change in stability as the bifurcation parameter r is varied. In general, at a Hopf bifurcation, as r passes through the bifurcation value r^*, there are three possible dynamics that may occur.

(i) At the bifurcation value r^* infinitely many neutrally stable concentric closed orbits encircle the equilibrium.

(ii) A stable spiral changes to a stable limit cycle for values of the parameter close to r^* (*supercritical bifurcation*).

(iii) A stable spiral and unstable limit cycle change to an unstable spiral for values of the parameter close to r^* (*subcritical bifurcation*).

Example 5.19 illustrates a change of stability of type (i). Figure 5.13 illustrates a supercritical and a subcritical bifurcation in x-y-r space. Stable solutions are identified by solid curves and unstable solutions by dashed curves.

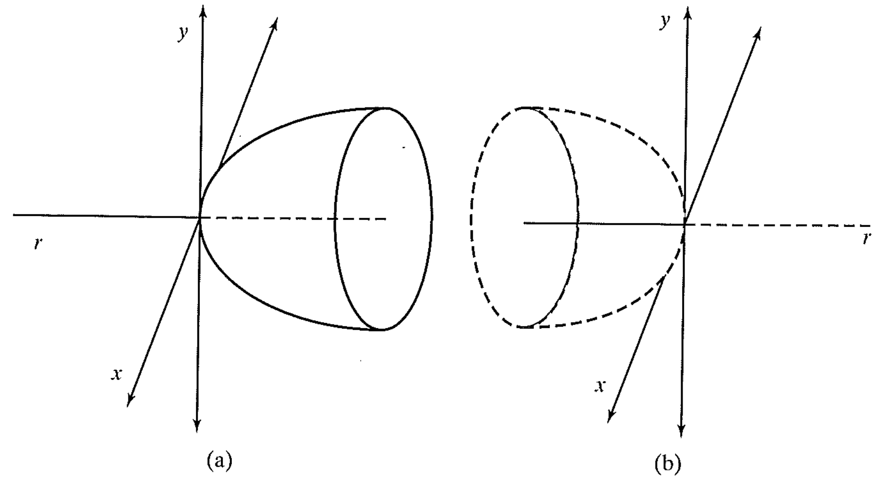

Figure 5.13 (a) Supercritical and (b) subcritical bifurcations in x-y-r space. Solid curves circling or on the r-axis are stable. Dashed curves are unstable.

The Hopf Bifurcation Theorem is stated as given by Hale and Koçak (1991). For a proof of this theorem see Hale and Koçak (1991) or Marsden and McCracken (1976). First the system is transformed so that the equilibrium is at the origin and the parameter r at $r^* = 0$ gives purely imaginary eigenvalues. System (5.10) is rewritten as follows:

$$\frac{dx}{dt} = a_{11}(r)x + a_{12}(r)y + f_1(x, y, r)$$

$$\frac{dy}{dt} = a_{21}(r)x + a_{22}(r)y + g_1(x, y, r). \tag{5.11}$$

The linearization of system (5.11) about the origin is given by $dZ/dt = J(r)Z$, where $Z = (x, y)^T$ and

$$J(r) = \begin{pmatrix} a_{11}(r) & a_{12}(r) \\ a_{21}(r) & a_{22}(r) \end{pmatrix} \tag{5.12}$$

is the Jacobian matrix evaluated at the origin.

Theorem 5.10 **(Hopf Bifurcation Theorem).** *Let f_1 and g_1 in system (5.11) have continuous third-order derivatives in x and y. Assume that the origin $(0, 0)$ is an equilibrium of (5.11) and that the Jacobian matrix $J(r)$, defined in (5.12), is valid for all sufficiently small $|r|$. In addition, assume that the eigenvalues of matrix $J(r)$ are $\alpha(r) \pm i\beta(r)$ with $\alpha(0) = 0$ and $\beta(0) \neq 0$ such that the eigenvalues cross the imaginary axis with nonzero speed (transversal),*

$$\left.\frac{d\alpha}{dr}\right|_{r=0} \neq 0.$$

Then, in any open set U containing the origin in $\mathbf{R}^2$ and for any $r_0 > 0$, there exists a value $\bar{r}$, $|\bar{r}| < r_0$ such that the system of differential equations (5.11) has a periodic solution for $r = \bar{r}$ in U (with approximate period $T = 2\pi/\beta(0)$). □

ample 5.20 Consider the linear system in Example 5.19 with bifurcation parameter r. We show that the conditions of the Hopf Bifurcation Theorem hold.

$$\frac{dx}{dt} = rx - y,$$

$$\frac{dy}{dt} = x + ry.$$

In this case, $f_1 = 0 = g_1$. The Jacobian matrix is

$$J(r) = \begin{pmatrix} r & -1 \\ 1 & r \end{pmatrix}.$$

with eigenvalues equal to $r \pm i$. Since $\alpha(r) = r$ and $\beta(r) = 1$, it follows that $\alpha(0) = 0$, $\beta(0) \neq 0$, and $d\alpha/dr = 1 \neq 0$. The conditions of the Hopf Bifurcation Theorem hold. In fact, we know that there exists a periodic solution for $r = 0$ in every neighborhood of the origin. ■

A computational method can be applied to determine whether a supercritical or subcritical bifurcation occurs. This method is given in the Appendix to this chapter.

ample 5.21 Consider the system

$$\frac{dx}{dt} = rx + y,$$

$$\frac{dy}{dt} = -x + ry - y^3. \tag{5.13}$$

There is an equilibrium at (0, 0). The Jacobian matrix is

$$J(r) = \begin{pmatrix} r & 1 \\ -1 & r - 3y^2 \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} r & 1 \\ -1 & r \end{pmatrix}.$$

The eigenvalues of $J(r)$ are $r \pm i$. The Hopf Bifurcation Theorem can be applied. In addition, a test for a supercritical or a subcritical bifurcation can be applied (Appendix) to show that the bifurcation at $r = 0$ is supercritical (Exercise 32). If the parameter r is sufficiently small and positive, then the system of differential equations has a stable periodic solution in a neighborhood of the origin. Figure 5.14 illustrates the dynamics of this system for $r = -1/2$ and $r = 1/2$. Note for $r = 1/2$ that there are three equilibria. ■

5.9 Delay Logistic Equation

The continuous logistic equation $dx/dt = rx(1 - x/K)$ has a discrete approximation which is given by the following difference equation: $x_{t+1} = x_t + rx_t(1 - x_t/K)$. This latter equation represents a delay in the growth, since the change in the population size does not occur until one unit later, t to $t + 1$. In the continuous logistic equation, the change in growth is instantaneous. As we have seen in Chapter 2, the equilibrium $\bar{x} = K$ can be destabilized in the difference equation model as r increases. The equilibrium $\bar{x} = K$ in the continuous logistic

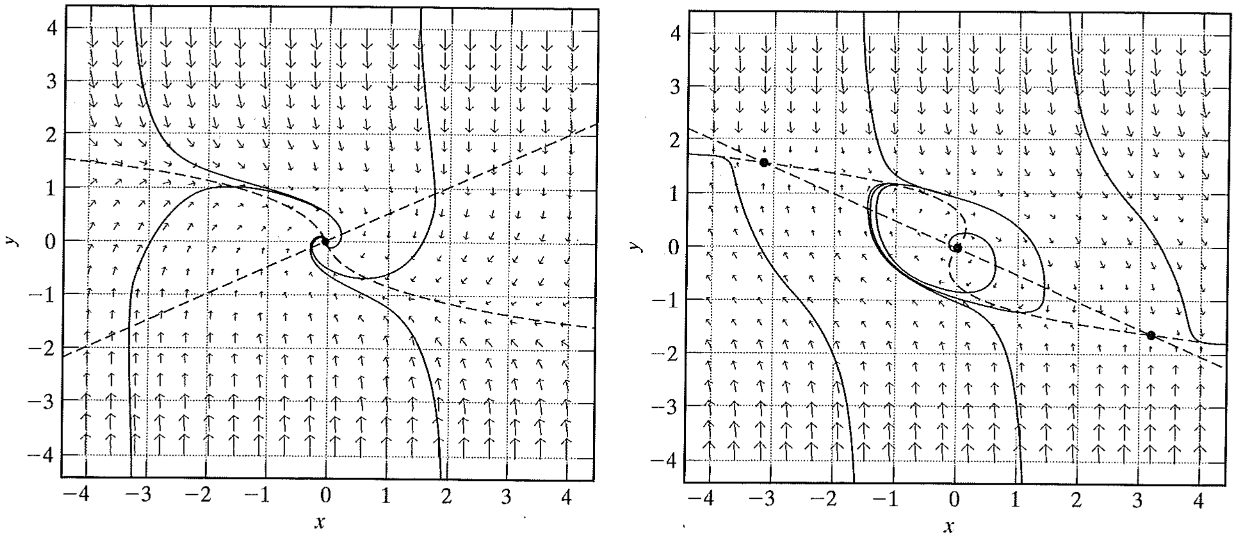

Figure 5.14 Dynamics of system (5.13) in the phase plane when $r = -1/2$ (left figure) and $r = 1/2$ (right figure). The equilibrium (0, 0) is stable in the figure on the left, but in the figure on the right solutions near the origin converge to a stable periodic solution.

equation is asymptotically stable if $r, K > 0$ and $x(0) > 0$. However, in the discrete case, the equilibrium $\bar{x} = K$ is only locally asymptotically stable if $0 < r < 2$. For values of r satisfying $r > 2$, solutions become periodic (period-doubling) and chaotic. Delays often change the stability of an equilibrium. We will show that putting a delay of length T in the density-dependent term in the continuous logistic equation changes the range of values r for which the equilibrium is stable.

Consider the logistic equation with a delay of T in the density-dependent factor,

$$\frac{dx\,(t)}{dt} = rx(t)\left(1 - \frac{x(t-T)}{K}\right). \tag{5.14}$$

The parameters $r, K,$ and T are positive. Parameter r is the intrinsic growth rate, K is the carrying capacity, and T is the delay parameter. Note that equation (5.14) still has two constant solutions or equilibria: $\bar{x} = 0$ and $\bar{x} = K$. The density-dependent factor, $1 - x(t-T)/K$, which regulates the rate of growth, is not instantaneous but depends on the population at an earlier time $t - T$. For example, the population size which affects food resources may not be immediately felt by the population but only after a period of time T. Equation (5.14) is sometimes referred to as the *Hutchinson-Wright equation* because it was first studied by the ecologist Hutchinson (1948) and the mathematician Wright (1946) (see also Kot, 2001). Note that to compute the solution to this discrete-delay equation for $t > 0$ it is necessary to know the value of $x(t)$ on the interval $[-T, 0]$. The method of steps can be applied to this delay model. To find the solution on the interval $[0, T]$, it is necessary to solve the following equation:

$$\frac{dx\,(t)}{dt} = rx(t)\left(1 - \frac{\phi_0(t-T)}{K}\right),$$

where $\phi_0(t) = x(t)$ on the interval $[-T, 0]$. We do not solve this differential equation by the method of steps, but determine the region of stability which depend on r and T, where the equilibrium K is locally asymptotically stable.